ACTIONS OF HIGHER–RANK ABELIAN GROUPS AND THEIR CONNECTIONS TO MODERN ANALYSIS AND NUMBER THEORY.

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- KAM method in rigidity (D.Damjanovich(Ph.D '04-PSU)–AK).
- Measure rigidity. Main problems and description of methods.(AK–R.Spatzier(Ph.D-'83-Warwick/Maryland); B.Kalinin(Ph.D-'00-PSU)–AK; M.Einsiedler(Post-Doc-01/02)–AK).
- Application to Littlewood conjecture (M.Einsiedler–E.Lindenstrauss–AK).

Connections with Moscow school and MGU

My MGU years 1960-68 (undergraduate-graduate); 1968-73 (part-time).

Both of my topics go back to Kolmogorov's fundamental contributions to modern analysis: KAM and entropy. Another root of 2-3: conditional measures (Rokhlin, Sinai)

Kolmolgorov was my mathematical "grandfather" through Sinai (and Ph. D. referee:"official opponent") My Ph. D. was connected with Kolmogorov's problems but to neither of the two topics.

I learned KAM from Arnol'd's lectures and articles (general outlines) and from Moser's aricles (techniques) especially from his Pisa lectures which I translated for Uspehi (1968).

I learned entropy and conditional measures from Sinai.

DIFFERENTIABLE GROUP ACTIONS.
Preliminaries

- M – differentiable manifold (usually compact).
- G – Lie group (usually connected or discrete).
- $\mathrm{Diff}^r(M)$ – the group of C^r diffeomorphisms of M;
$$r = 0, 1, 2, \ldots; \infty, \omega.$$
- A continuous homomorphism $\alpha : G \to \mathrm{Diff}^r(M)$ is called a C^r*–action* of G on M.

We give general definitions but we will discuss in the talk only the actions of abelian groups: $\mathbb{Z}^k$ and $\mathbb{R}^k$. Our substantive results deal with the *higher rank* case ($k \geq 2$; commuting maps and vector–fields); the classical cases of $\mathbb{Z}$ actions (iterates of a single map) and $\mathbb{R}$ actions (flows generated by ODE) will appear mostly for comparison and contrast.

C^r–conjugacy, orbit equivalence, time change

- G-actions α and β are C^r–*conjugate* if $\alpha = h \circ \beta \circ h^{-1}$ for some $h \in \mathrm{Diff}^r(M)$.
- G-actions α and β are C^r–*conjugate up to an automorphism* if $\alpha \circ \rho = h \circ \beta \circ h^{-1}$ for some $h \in \mathrm{Diff}^r(M)$ and an automorphism ρ of G.
- For G continuous let $\mathcal{O}_\alpha$ be the orbit foliation of α.
- α and β are C^r–*orbit equivalent* if $O_\alpha = hO_\beta$ for some $h \in \mathrm{Diff}^r(M)$.
- β is a *time change* of α if $O_\alpha = O_\beta$.
- Actions are orbit equivalent iff one is conjugate to a time change of the other.
- Time change is determined by a certain G–valued 1–*cocycle* over the action.

Rigidity

Let $0 \leq k \leq m \leq \ r$. (Assume $\omega > \infty > n$ for any natural n.)

- For discrete G a C^r action α of G is C^m *k–rigid* if any C^r action β sufficiently close to α in C^m topology is C^k conjugate to α.
- For continuous G a C^r action α of G is C^m *k–rigid* if any C^r action β sufficiently close to α in C^m topology is C^k conjugate to α up to an automorphism.
- C^m *k–orbit rigidity* defined similarly with C^k orbit equivalence instead of conjugacy.
- C^m 0–orbit rigidity is called C^m *structural stability.*

Usually it is also assumed that the conjugacy or orbit equivalence is C^k close to identity.

The classical cases: $G = \mathbb{Z}, \mathbb{R}$

- C^1 structural stability completely characterized; closely connected with (uniform) *hyperbolicity*.
- In the structurally stable continuous time case there are infinitely many *moduli* of C^0 (topological) conjugacy; closely related to lengths of periodic orbits.
- Existence of C^m (but not C^1) structural stability for $m > 1$ is unknown; it is unlikely, but the proof is currently beyond reach due to the lack of understanding of *global C^m perturbation constructions* for $m \geq 2$.
- C^m 1–rigidity or orbit rigidity is impossible due to the good understanding of *local C^1 perturbations*.
- C^m k–rigidity or orbit rigidity is highly unlikely and can be ruled out in many cases (e.g. on any surface other than torus).

An example: Structural stability of a hyperbolic toral automorphism

Such map F_A on the torus $\mathbb{T}^N$ is given by an integer $N \times N$ matrix A, $\det A = \pm 1$. Assume A has no eigenvalues of absolute value 1 then F_A is $\mathcal{C}^1$ structurally stable. Namely, if g is a $\mathcal{C}^1$-small perturbation of F_A then there exists a homeomorphism H of the torus such that:

$$g \circ H = H \circ F_A \tag{1}$$

It is enough to find a continious bounded 1-periodic map h on $\mathbb{R}^N$ such that:

$$h \circ A - A \circ h = \tilde{g} \circ (id + h) \tag{2}$$

where A and $A + \tilde{g}$ are lifts of F_A and g to $\mathbb{R}^N$. Let

$$\mathcal{L}(h) \stackrel{\text{def}}{=} h \circ A - A \circ h, \text{ and } \mathcal{N}(h) \stackrel{\text{def}}{=} \tilde{g} \circ (id + h). \tag{3}$$

Clearly $\|\mathcal{N}(h) - \mathcal{N}(h')\| \leq \|D\tilde{g}\| \|h - h'\|$, thus if $\mathcal{L}^{-1}$ is a bounded linear operator on the space B of continuous N-periodic maps than $\|D\tilde{g}\|$ being sufficiently small assures that the operator $\mathcal{L}^{-1}\mathcal{N}$ is a contraction on the Banach space B. Thus it has a unique fixed point h which by (2) gives a continuous surjective map $H = id + h$ satisfying (1). The operator $\mathcal{L}$ has bounded inverse due to hyperbolicity. Assuming $A$is diagonalizable take an eigen-basis: the equation (3) splits: $\mathcal{L}_i(h_i) \stackrel{\text{def}}{=} h_i \circ A - \lambda_i h_i$ where λ_i are eigenvalues of A and h_i coordinate functions of h. Each of these equations can be inverted as:

$$\mathcal{L}_i^{-1}(h_i) = -\sum_{n=0}^{\infty} \lambda_i^{-(n+1)} h_i \circ A^n \quad \text{for} \quad |\lambda_i| > 1$$

$$\mathcal{L}_i^{-1}(h_i) = \sum_{n=0}^{\infty} \lambda_i^{n} h_i \circ A^{-(n+1)} \quad \text{for} \quad |\lambda_i| < 1$$

and the resulting operator is clearly bounded on the space B;H is injective since A is expansive. Thus H conjugates g and F_A.

The higher rank case

The differentiable orbit structure of smooth actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$ for $k > 1$, is remarkably different from the classical cases $k = 1$. While differentiable rigidity as well as local classification by finitely many moduli is most likely impossible in the classical cases, these phenomena appear in the higher–rank case.

This was shown in the nineties for most standard hyperbolic actions, such as actions by *hyperbolic automorphisms of the torus, Weyl Chamber flows* and other hyperbolic homogeneous actions in a series of papers by R. Spatzier, M. Guysinsky and the speaker.

The methods there combine classical hyperbolic fixed point techniques (Hirsch–Pugh–Shub theory) to establish structiral stability, with the theory of nonstationary normal forms and cocycle (parameter) rigidity to show transversal regularity and rigidity of time changes correspondingly.

A new progress has been achieved recently with the introduction of a KAM type iteration method where the cocycle rigidity serves as an inductive step to invert the linearized equation. These scheme has been carried out jointly with my Ph.D. student Danijela Damjanovic in the case of commuting PARTIALLY HYPERBOLIC automorphisms of the torus.

Notice that individual element ARE NOT STRUCTURALLY STABLE here so the method based on a priori estimates is not sufficient.

The work on the semisimple case is in progress.

In the earlier version a mixed method was used.

- First C^0 stability of the neutral foliation was established using Hirsch–Pugh–Shub theory.
- Then regularity of the conjugacy transversally to the neutral direction was proved by the nonstationary normal forms method.
- And finally KAM scheme was applied to perturbations in the neutral direction only ;
- Cocycle rigidity is used to solve the linearized equation for an iterative step.

In the definitive version

Danijela Damjanovic and Anatole Katok. *Local Rigidity of Partially Hyperbolic Actions on the Torus*, http://www.math.psu.edu/katok_a/papers.html

KAM method completely takes over; this is Damjanovich's original contribution. The proof for hyperbolic and partially hyperbolic cases is unified. In particular, non-trivial Jordan blocks previously excluded in the hyperbolic case are covered now.

KAM METHOD

• M – compact smooth manifold;

• $g : M \to M$ – a "model" (linear, algebraic, homogeneous, etc);
Assume linear structure in a space of diffeomorphisms near g and near Id;

• $f = g + u$ – a "perturbation" (nonlinear, general).
Want to show that f is (smoothly) conjugate to g via unknown $h = \mathrm{Id} + w$.

• Conjugacy equation as an implicit-function problem:

$$g = \mathcal{F}(f, h) := h^{-1} \circ f \circ h. \tag{4}$$

• The "group property":

$$\mathcal{F}(f, \varphi \circ \psi) = \mathcal{F}(\mathcal{F}(f, \varphi), \psi), \qquad \mathcal{F}(f, \mathrm{Id}) = f. \tag{5}$$

Linearized equation.

- $D_1\mathcal{F}$ and $D_2\mathcal{F}$ partial differentials with respect to f.
- First order Taylor expansion of (4) at (g, Id):

$$\mathcal{F}(f, h) = \mathcal{F}(g, \mathrm{Id}) + D_1\mathcal{F}(g, \mathrm{Id})(u) + D_2\mathcal{F}(g, \mathrm{Id})(w) + \mathcal{R}(f, h);$$

$\mathcal{R}(f, h)$ is of second order in (u, w).

- If h solves the linearized equation (obtained by dropping $\mathcal{R}$), then

$$\mathcal{F}(g, \mathrm{Id}) + D_1\mathcal{F}(g, \mathrm{Id})(f - g) + D_2\mathcal{F}(g, \mathrm{Id})w = g. \tag{6}$$

Since $\mathcal{F}(\cdot, \mathrm{Id}) = \mathrm{Id}(\cdot)$ by (5), $D_1\mathcal{F}(g, \mathrm{Id}) = \mathrm{Id}$, and

$$u + D_2\mathcal{F}(g, \mathrm{Id})w = 0. \tag{7}$$

- If $D_2\mathcal{F}(g, \mathrm{Id})$ is invertible, then $w = -\left(D_2\mathcal{F}(g, \mathrm{Id})\right)^{-1} u$.

Quadratic convergence

• In the case of invertibility, w is of the same order as u, and substituting $h = \mathrm{Id} + w$ into $\mathcal{F}(f, h)$ we obtain a function $f_1 = h^{-1} \circ f \circ h = \mathcal{F}(f, h) = g + \mathcal{R}(f, h)$, so the size of $u_1 = f_1 - g = \mathcal{R}(f, h)$ is formally of second order in the size of $u = f - g$.

• **Iterative process.** Assuming that $f_1, \ldots, f_n$ have been constructed, solve the equation

$$f_n - g + D_2\mathcal{F}(g, \mathrm{Id}) w_{n+1} = 0$$

and set

$$h_{n+1} = h_n \circ (\mathrm{Id} + w_{n+1}) \text{ and } f_{n+1} = (\mathrm{Id} + w_{n+1})^{-1} \circ f_n \circ (\mathrm{Id} + w_n).$$

• To justify the iterative process and prove convergence, one needs to estimate the difference between $\mathcal{F}$ and its linearization near (g, Id).

Intrinsic subtlety of the conjugacy problem

Notice that at every step the linear part is inverted at (g, Id), rather than at the intermediate points as in the elementary Newton method. This is the main reason which makes analytic difficulties manageable (however often still quite formidable):

- There are usually *obstructions* to solvability of the linearized equation: the operator $D_2\mathcal{F}(g, \mathrm{Id})$ is only invertible at the kernel of these obstructions which may have finite or infinite codimension.
- Even at the kernel there is usually no bounded inverse in any natural class of regularity (C^r, Sobolev, analytic in a fixed domain).

Recall the notions of rigidity (and slightly change notations for convenience)

An action α of a finitely generated group A on manifold M is $\mathcal{C}^{k,r,l}$ *locally rigid* if any sufficiently $\mathcal{C}^r$-small $\mathcal{C}^k$ perturbation $\tilde{\alpha}$ is $\mathcal{C}^l$ conjugate to α, i.e there exists a $\mathcal{C}^l$ diffeomorphism $\mathcal{H}$ of M which conjugates $\tilde{\alpha}$ to α: $\mathcal{H} \circ \alpha(g) = \tilde{\alpha}(g) \circ \mathcal{H}$ for all $g \in A$. $\mathcal{C}^{\infty,1,\infty}$ is usually refered to as $\mathcal{C}^\infty$-*local rigidity.*

Theorem 1 *Let $\alpha : \mathbb{Z}^d \times \mathbb{T}^N \to \mathbb{T}^N$ be a $\mathcal{C}^\infty$ partially hyperbolic action of $\mathbb{Z}^d$ $(d \geq 2)$ by toral automorphisms. Let $\tilde{\alpha} : \mathbb{Z}^d \times \mathbb{T}^N \to \mathbb{T}^N$ be a $\mathcal{C}^l$-small $\mathcal{C}^\infty$ perturbation of α. Assume that the action α has no non-trivial rank-one factors. Then there exists a $\mathcal{C}^\infty$ map $H : \mathbb{T}^N \to \mathbb{T}^N$ such that $\alpha \circ H = H \circ \tilde{\alpha}$, i.e α is $\mathcal{C}^{\infty,l,\infty}$ locally rigid.*

Here l depends on the dimension of the torus and the linear action.

The scheme of proof:

α - linear action

$\tilde{\alpha}$ - perturbation

Let $\mathcal{R} = \tilde{\alpha} - \alpha$ be "small" $(\|(R)\|_{\mathcal{C}^0} \leq \varepsilon, \|(R)\|_{\mathcal{C}^l} \leq \varepsilon^{-1}, \varepsilon$ small)

The goal is to show the existence of $H : \mathbb{T}^N \to \mathbb{T}^N$ such that $\tilde{\alpha} \circ H = H \circ \alpha$, $H = \mathrm{id} + \Delta$ where Δ should be "small" as well.

In terms of Δ we need

$$\alpha\Delta - \Delta \circ \alpha = -\mathcal{R} \circ (\mathrm{id} + \Delta)$$

If Δ is a solution for the corresponding linearized equation

$$\alpha\Delta - \Delta \circ \alpha = -\mathcal{R}$$

then

$$\tilde{\alpha}^{(1)} \stackrel{\mathrm{def}}{=} H^{-1} \circ \tilde{\alpha} \circ H$$

should be "quadratically" close to α i.e.

$$\mathcal{R}^{(1)} \stackrel{\text{def}}{=} \tilde{\alpha}^{(1)} - \alpha$$

should be "quadratically small" with respect to $\mathcal{R}$.

It is easy to see that the new error is is:

$$\mathcal{R}^{(1)} = \tilde{\alpha}^{(1)} - \alpha = \Big[\Delta \circ \tilde{\alpha}^{(1)} - \Delta \circ \alpha + \mathcal{R}(\mathrm{id} + \Delta) - \mathcal{R}\Big] +$$

$$\Big[\mathcal{R} - [\alpha \circ \Delta - \Delta \circ \alpha]\Big] .$$

The part of the error in the first parentheses is easy to estimate providing Δ is "small"

Therefore, it is enough to solve the linearized equation approximately (i.e. - with an error "quadratically small" w.r.t. $\mathcal{R}$) in order to run the KAM iteration scheme and produce a solution in the limit.

It is enough to produce a conjugacy H for two generators. The same conjugacy will work (due to commutativity and ergodicity assumptions) for all other elements of the action.

Therefore, solving the linearized equation

$$\alpha\Delta - \Delta \circ \alpha = -\mathcal{R}$$

reduces to

$$A\Delta - \Delta \circ A = -\mathcal{R}_A, \quad B\Delta - \Delta \circ B = -\mathcal{R}_B \tag{8}$$

where $A \stackrel{\text{def}}{=} \alpha(g_1)$, $B \stackrel{\text{def}}{=} \alpha(g_2)$ are two $N \times N$ matrices with no roots of of unity in the spectrum.

It is possible to solve the linearized equation (8) providing

$$L(\mathcal{R}_A, \mathcal{R}_B) \stackrel{\text{def}}{=} (\mathcal{R}_A \circ B - B\mathcal{R}_B) - (\mathcal{R}_B \circ A - A\mathcal{R}_B) = 0$$

with fixed loss of the regularity in $\mathcal{C}^\infty$ case

Even if $\mathcal{R}_A$ and $\mathcal{R}_B$ do not satisfy the solvability condition above, it is still possible to approximate both by maps which satisfy the solvability condition with error bounded by the size of $L(\mathcal{R}_A, \mathcal{R}_B)$. Again with fixed loss of regularity in the $\mathcal{C}^\infty$ case.

We show that if $\tilde{\alpha} = \alpha + \mathcal{R}$ is a commutative action then $L(\mathcal{R}_A, \mathcal{R}_B)$ is "quadratically small" w.r.t $\mathcal{R}_A$, $\mathcal{R}_B$.

From these we get approximate solution to the linearized equation (8).

Existence of non-trivial examples

Theorem 2 *Genuinely partially hyperbolic (i.e ergodic, not hyperbolic)* $\mathbb{Z}^2$ *actions by toral automorphisms exist: on any torus of even dimension* $N \geq 6$ *there are irreducible examples while on any torus of odd dimension* $N \geq 9$ *there are only reducible examples. There are no examples on tori of dimension* $N \leq 5$ *and* $N = 7$.

An explicit example in dimension 6 (S.Katok)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2 \\ -2 & 4 & 4 & 0 & 7 & -2 \\ 2 & -6 & -6 & -2 & -10 & 3 \\ -3 & 8 & 9 & 3 & 13 & -4 \\ 4 & -11 & -12 & -3 & -17 & 5 \\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}$$

- Existence in even dimensions $N \geq 6$ is related to recurrent polynomials and symplectic structures.
- In odd dimensions even individual irreducible genuinely partially hyperbolic automorphisms do not exist.
- In dimension four (and of course two) there is not enough room for a $\mathbb{Z}^2$ partially hyperbolic action since at least three hyperbolic directions are needed.
- For odd $N \geq 9$ take a product of an irreducible example in dimension six and hyperbolic action in dimension $N - 6$
- Dimension seven is excluded because the only split would be $3 + 4$.