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**The development of dynamics in the 20th century and the contribution of Jürgen Moser.**  
(English summary)

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J. K. Moser (1928–1999) made profound and fundamental contributions to many branches of mathematics. The present paper surveys his work in dynamics and related areas. For brief accounts of all the achievements of Moser in mathematics, the authors recommend the notes by E. J. Zehnder [Jahresber. Deutsch. Math.-Verein. **95** (1993), no. 2, 85–94; [MR1218340 \(94d:01080\)](#)] and J. N. Mather et al. [Notices Amer. Math. Soc. **47** (2000), no. 11, 1392–1405; [MR1794131 \(2001h:01054\)](#)].

The first part of the paper outlines the development of dynamics from Newton and Laplace to the contemporary theories, with an emphasis on the 20th century, and provides general characteristics of the topics and style of Moser’s work. Here are some quotations: “Always keenly interested in the work of others, he was able to discern the fundamental trends and invariably made essential, often fundamental, contributions.” “We cannot think of another mathematician in the period after 1960 who had such a broad view and comprehensive understanding of virtually all major trends in dynamics and influenced their development to a similar degree.” “In his work he usually searched for wisdom rather than simply knowledge, and thus he strongly emphasized developments of methods and insights over pushing a specific result to the limit.” “The leading theme of virtually all of Moser’s work in dynamics is the search for elements of stable behavior in dynamical systems with respect to either initial conditions or perturbations of the system.” In the second part of the paper, the authors discuss (unavoidably omitting many details) Moser’s contributions to the KAM (Kolmogorov-Arnol’d-Moser) theory, the Aubry-Mather theory, completely integrable Hamiltonian systems, and hyperbolic dynamics, as well as some separate results. The exposition is very vivid and provides both the mathematical ideas and historical accounts. However, it is hardly intelligible to a layman. The bibliography contains 91 references, including 29 works by Moser.

Reviewed by *Mikhail B. Sevryuk*

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