MULTIPLICATIVE PROPERTIES OF AUTOMORPHISMS

ANATOLE KATOK

1. Main Inequalities

Suppose that an automorphism T admits a cyclic approximation by periodic transformations with some speed. Functions that are constant on the elements of partition ξ_n form a q_n-dimensional space H_n, which is obviously invariant under the operator U_{T_n}. The spectrum of the operator U_{T_n} in H_n is simple and consists of eigenvalues $\lambda_{n,j} = e^{\frac{2\pi i}{q_n}}$, $j = 0, \dots, q_n - 1$. Denote by $h_{n,j}$ a normalized eigenvector corresponding to the eigenvalue $\lambda_{n,j}$. Functions $h_{n,j}$ are defined uniquely up to complex multiplier of modulus 1. The following normalization of $h_{n,j}$ will be useful for us. Let $c \in \xi_n$. Set $h_{n,i}(x) = e^{\frac{2\pi ijk}{q_n}}$ if $x \in T_n^k c$, $k = 0, \dots, q_n - 1$. By Δ_n^j we will denote the arc of the unit circle $(e^{\frac{2\pi ij-\pi i}{q_n}}, e^{\frac{2\pi ij-\pi i}{q_n}})$, and $E(\Delta)$ $(\Delta \subset S^1)$ will denote the projectors from the spectral family of the operator U_T.

Theorem 1.1. *Let an automorphism T admits a cyclic approximation by periodic transformations with speed $f(n)$, $g(x) \in L_2(M, \mu)$, $\|g(x)\| = 1$, $g_n(x)$ is a projection of the function $g(x)$ to the subspace H_n. Then*

$$\sum_{j=0}^{q_n-1} \left\| E(\Delta_n^j) g_n - (g_n, h_{n,j}) h_{n,j} \right\| < 2\sqrt{2} q_n^2 \sqrt{f(q_n)}, \tag{1.1}$$

$$\sum_{j=0}^{q_n-1} \left\| E(\Delta_n^j) g_n - (g_n, h_{n,j}) h_{n,j} \right\|^2 < 4 q_n^3 f(q_n). \tag{1.2}$$

Proof. Set $h'_{n,j} := h_{n,j} - E(\Delta_n^j) h_{n,j}$. We eill evaluate $\|h'_{n,j}\|$. Outside of the set $E_n = \bigcup_{i=1}^{q_n} (T^{-1} c_{n,j} \triangle T_n^{-1} c_{n,j})$ we have $U_T h_{n,j} = \lambda_{n,j} h_{n,j}$. Therefore

$$|(U_T h_{n,j}, h_{n,j}) - \lambda_{n,j}| \le 2\mu(E_n) < 2f(q_n). \tag{1.3}$$

Let $\|h'_{n,j}\| = a$. Then

$$|(U_T h_{n,j}, h_{n,j}) - \lambda_{n,j}| \ge \frac{a^2}{q_n^2}. \tag{1.4}$$

It follows from (1.3) and (1.4) that

$$\frac{a^2}{q_n^2} < 2f(q_n), \quad \|h'_{n,j}\| < \sqrt{2q_n^2 f(q_n)}. \tag{1.5}$$

For $g_n(x)$ there are two orthogonal decompositions:

$$g_n = \sum_{j=0}^{q_n-1} E(\Delta_n^j) g_n = \sum_{j=0}^{q_n-1} (g_n, h_{n,j}) h_{n,j}. \tag{1.6}$$

Substituting the equation $h_{n,j} = E(\Delta_n^j)h_{n,j} + h'_{n,j}$ in (1.6), we obtain

$$\sum_{j=0}^{q_n-1} E(\Delta_n^j)\left(g_n - (g_n, h_{n,j})h_{n,j}\right) = \sum_{j=0}^{q_n-1} (g_n, h_{n,j})h'_{n,j}.$$

Since the functions $h_{n,j}$ are orthonormal and $\|g_n\| \le 1$, then $\sum_{j=0}^{q_n-1} |(g_n, h_{n,j})| < \sqrt{q_n}$ and hence

$$\Big\| \sum_{j=0}^{q_n-1} E(\Delta_n^j)(g_n - (g_n, h_{n,j})h_{n,j})\Big\| \le \sqrt{2q_n^3 f(q_n)}.$$

The vectors $H_{n,j} := E(\Delta_n^j)(g_n - (g_n, h_{n,j})h_{n,j})$ are also orthogonal and hence

$$\sum_{j=0}^{q_n-1} \|H_{n,j}\|^2 = \Big\| \sum_{j=0}^{q_n-1} H_{n,j}\Big\|^2, \quad \sum_{j=0}^{q_n-1} \|H_{n,j}\| \le \sqrt{q_n}\Big\| \sum_{j=0}^{q_n-1} H_{n,j}\Big\|.$$

Now it is easy to obtain inequalities (1.1) and (1.2):

$$\begin{aligned}\sum_{j=0}^{q_n-1} \big\|E(\Delta_n^j)g_n - (g_n, h_{n,j})h_{n,j}\big\| &= \sum_{j=0}^{q_n-1} \big\|H_{n,j} - (g_n, h_{n,j})h'_{n,j}\big\| \\ &\le \sum_{j=0}^{q_n-1} \|H_{n,j}\| + \sum_{j=0}^{q_n-1} \|(g_n, h_{n,j})h'_{n,j}\| \\ &\le \sqrt{2q_n^4 f(q_n)} + \sqrt{2q_n^3 f(q_n)} \\ &\le 2\sqrt{2}q_n^2\sqrt{f(q_n)},\end{aligned}$$

$$\begin{aligned}\sum_{j=0}^{q_n-1} \big\|E(\Delta_n^j)g_n - (g_n, h_{n,j})h_{n,j}\big\|^2 &= \sum_{j=0}^{q_n-1} \big\|H_{n,j} - (g_n, h_{n,j})h'_{n,j}\big\|^2 \\ &= \sum_{j=0}^{q_n-1} \|H_{n,j}\|^2 + \sum_{j=0}^{q_n-1} \|(g_n, h_{n,j})h'_{n,j}\|^2 \\ &\le 4q_n^3 f(q_n).\end{aligned}$$

□

2. Corollaries from Main Inequalities

Let Δ be an open arc of the unit circle. Denote by Δ_n the union of all the arcs Δ_n^j containing in Δ. Denote by J_n the set of all j for which $\Delta_n^j \subset \Delta_n$. Let $g \in L_2(M, \mu)$. Consider the function $G_n(x) := \Sigma_{j\in J_n} |E(\Delta_n^j)g(x)|^2$. By $\|\cdot\|_1$ we will denote the norm in $L_1(M, \mu)$, and as before $\|\cdot\|$ will denote the norm in $L_2(M, \mu)$.

Theorem 2.1. *Let an automorphism T admits a cyclic approximation by periodic transformations with speed $o(n^{-4})$ and let a function g be such that*

$$\|g - g_n\| = o\left(\frac{1}{q_n}\right), \tag{2.1}$$

where g_n is a projection of g on the space H_n. Then

$$\|G_n(x) - (E(\Delta)g, g)\|_1 \to 0.$$

Remark 2.2. Functions g that satisfy the condition (2.1) exist. Moreover, they form a comeagre set in $L_2(M,\mu)$ and among them there exist vectors of maximal spectral type in $L_2(M,\mu)$.

Remark 2.3. It T is an ergodic automorphism with discrete spectrum then the analog of Theorem 2.1 is true for some class of functions and follows from the fact that the modulus of an invariant function is constant. Ya. Fomin have proven the analogous statement (with convergence in L_2) for a wider class of automorphisms satisfying so called condition A. In our case, restrictions to g do not imply that $G_n \in L_2(M,\mu)$, so we have to prove the convergence in $L_1(M,\mu)$. Note that Theorem 2.1 follows from inequality (1.1), which shows that for 'most' of j function $E(\Delta_n^j)g_n$ (which is 'almost δ-function' on the spectrum) is closed to a function with constant modulus.

Proof of Theorem 2.1. Introduce the function $G'_n(x) := \sum_{j\in J_n} |(g_n,h_{n,j})h_{n,j}(x)|^2$. Since $|h_{n,j}(x)|=1$, then $G'_n(x) = \sum_{j\in J_n} |(g_n,h_{n,j})|^2$. We have

$$\begin{aligned}\|G_n - G'_n\|_1 &= \Big\| \sum_{j\in J_n} |E(\Delta_n^j)g|^2 - |(g,h_{n,j})|^2 \Big\|_1 \\ &\le \sum_{j\in J_n} \big\| |E(\Delta_n^j)g|^2 - |(g,h_{n,j})|^2 \big\|_1 \\ &\le \sum_{j\in J_n} \big\| |E(\Delta_n^j)g|^2 - |E(\Delta_n^j)g_n|^2 \big\|_1 \\ &+ \sum_{j\in J_n} \big\| |E(\Delta_n^j)g_n|^2 - |(g_n,h_{n,j})h_{n,j}|^2 \big\|_1 .\end{aligned}$$

It is clear that $(g,h_{n,j}) = (g_n,h_{n,j})$. Set $\|g-g_n\| = \frac{a_n}{q_n}$. By the assumption of the theorem, $a_n \to 0$. Then

$$\begin{aligned}&\big\| |E(\Delta_n^j)g|^2 - |E(\Delta_n^j)g_n|^2 \big\|_1 \le \\ &\le \big\| |E(\Delta_n^j)g| - |E(\Delta_n^j)g_n| \big\| \big\| |E(\Delta_n^j)g| + |E(\Delta_n^j)g_n| \big\| \le \\ &\le 2\|E(\Delta_n^j)(g-g_n)\| \le 2\|g-g_n\| = \frac{2a_n}{q_n}.\end{aligned}$$

In a similar way,

$$\begin{aligned}&\big\| |E(\Delta_n^j)g_n|^2 - |(g_n,h_{n,j})h_{n,j}|^2 \big\|_1 \le \\ &\le \big\| |E(\Delta_n^j)g_n| - |(g_n,h_{n,j})h_{n,j}| \big\| \big\| |E(\Delta_n^j)g_n| + |(g_n,h_{n,j})h_{n,j}| \big\| \le \\ &\le 2\|E(\Delta_n^j)g_n - (g_n,h_{n,j})h_{n,j}\|.\end{aligned}$$

Hence,

$$\|G_n - G'_n\|_1 \le 2a_n + 2\sum_{j\in J_n} \big\|E(\Delta_n^j)g_n - (g_n,h_{n,j})h_{n,j}\big\|.$$

It follows from (1.1) that $\|G_n - G'_n\| \to 0$. It remains to show that $G'_n(x) \to (E(\Delta)g,g)$ (i. e. $G_n(x)$ is constant). Clearly, $(E(\Delta_n)g,g) \to (E(\Delta)g,g)$ and $|(E(\Delta_n)g,g) - (E(\Delta_n)g_n,g_n)| \to 0$. We have

$$\begin{aligned}(E(\Delta_n)g_n,g_n) &= \|E(\Delta_n)g_n\|^2, \\ G'_n &= \Big\| \sum_{j\in J_n} |(g_n,h_{n,j})h_{n,j} \Big\|^2,\end{aligned}$$

$$|G'_n - (E(\Delta_n)g_n, g_n)| \le 2\|g_n\| \Big\| \sum_{j\in J_n} \big(E(\Delta_n^j)g_n - (g_n, h_{n,j})h_{n,j}\big) \Big\| \le 2 \sum_{j\in J_n} \big\|E(\Delta_n^j)g_n - (g_n, h_{n,j})h_{n,j}\big\|.$$

The latest statement tends to 0 by Theorem 1.1 □

Theorem 2.4. *If an automorphism T admits a cyclic approximation by periodic transformations with speed $o(n^{-3})$ then the operator U_T has simple spectrum.*

Proof. Suppose that the operator U_T does not have simple spectrum. Then there exist two functions $g^{(1)}, g^{(2)} \in L_2(M, \mu)$, $\|g^{(1)}\| = \|g^{(2)}\| = 1$, such that for any $\Delta \subset S^1$, $\|E(\Delta)g^{(1)}\| = \|E(\Delta)g^{(2)}\|$ and $(E(\Delta)g^{(1)}, E(\Delta)g^{(2)}) = 0$. Let $H_{n,j}$ stands for a 1-dimensional subspace spanned by the vector $h_{n,j}$. Since the vectors $E(\Delta_n^j)g^{(1)}$ and $E(\Delta_n^j)g^{(2)}$ are orthogonal and have the same norm, then

$$\rho^2(E(\Delta_n^j)g^{(1)}, H_{n,j}) + \rho^2(E(\Delta_n^j)g^{(2)}, H_{n,j}) \ge \|E(\Delta_n^j)g^{(1)}\|^2.$$

On the other hand,

$$\rho^2(E(\Delta_n^j)g^{(i)}, H_{n,j}) \le 2\big(\|E(\Delta_n^j)(g^{(i)} - g_n^{(i)})\|^2 + \|E(\Delta_n^j)g_n^{(i)} - (g_n^{(i)}, h_{n,j})h_{n,j}\|^2\big),$$

where $i = 1, 2$, i. e.

$$\|E(\Delta_n^j)g^{(1)}\|^2 \le 2\big(\|E(\Delta_n^j)(g^{(1)} - g_n^{(1)})\|^2 + \|E(\Delta_n^j)(g^{(2)} - g_n^{(2)})\|^2 + \|E(\Delta_n^j)g_n^{(1)} - (g_n^{(1)}, h_{n,j})h_{n,j}\|^2 + \|E(\Delta_n^j)g_n^{(2)} - (g_n^{(2)}, h_{n,j})h_{n,j}\|^2\big). \tag{2.2}$$

Summing inequalities (2.2) for $j = 0, \ldots, q_n - 1$, we get

$$1 = \|g^{(1)}\|^2 \le 2\Big(\|g^{(1)} - g_n^{(1)}\|^2 + \|g^{(2)} - g_n^{(2)}\|^2 + \sum_{j=0}^{q_n-1} \big\|E(\Delta_n^j)g_n^{(1)} - (g_n^{(1)}, h_{n,j})h_{n,j}\big\|^2 + \sum_{j=0}^{q_n-1} \big\|E(\Delta_n^j)g_n^{(2)} - (g_n^{(2)}, h_{n,j})h_{n,j}\big\|^2\Big). \tag{2.3}$$

By inequality (1.2), the statement in the right side of (2.3) tends to 0. This contradiction concludes the proof. □

Remark 2.5. The assertion of Theorem 3.3 is also true under weaker conditions on the speed of cyclic approximation by periodic transformations (see Theorem 3.1 from [12]). We have given the proof of Theorem 3.3 to illustrate an application of inequalities from Section 1.

3. Multiplication formula

Suppose that an automorphism T has simple spectrum, $g \in L_2(M, \mu)$ is a cyclic vector, $\|g\| = 1$. Then the operator U_T is isomorphic to the operator of multiplication by λ in $L_2(S^1, \sigma)$. Moreover, this isomorphism can be chosen in such a way that g maps to 1. We will call this isomorphism the *spectral representation corresponding to vector g*. Let functions $F(x), G(x) \in L_2(M, \mu)$ map by the spectral representation corresponding to vector g to functions $\varphi(\lambda)$ and $\gamma(\lambda)$, and function $F(x) \cdot G(x)$ maps to a function $\rho^{\varphi,\gamma}(\lambda)$. In this section we will show that for sufficiently large speed of approximation of the automorphism T and under certain assumption on g, F and G, $\rho^{\varphi,\gamma}(\lambda)$ can be represented as a limit of a sequence of functions $\rho_n^{\varphi,\gamma}(\lambda)$, where $\rho_n^{\varphi,\gamma}(\lambda)$ is defined by $\varphi(\lambda)$, $\gamma(\lambda)$, the spectral measure

σ and the inner products of function g with eigenfunctions $h_{n,j}$ of operators U_{T_n}. As before, inequality (1.1) will be the main tool for evaluations. Now pass to the precise formulation.

Theorem 3.1. *Suppose that an automorphism T admits a cyclic approximation by periodic transformations with speed $o(n^{-8})$, a function g is bounded, $\|g\| = 1$ and $\|g - g_n\| = o(q_n^{-\frac{5}{2}})$. Let $F(x), G(x) \in L_2(M, \mu)$ be bounded functions with bounded spectral representations $\varphi(\lambda)$ and $\gamma(\lambda)$. Then the spectral representation of the function $F(x) \cdot G(x)$ has the form $\rho^{\varphi,\gamma}(\lambda) = \lim_{n\to\infty} \rho_n^{\varphi,\gamma}(\lambda)$, where*

$$\rho_n^{\varphi,\gamma}(\lambda) = \begin{cases} \sum\limits_{\substack{j_1+j_2=j \\ j_1,j_2\in J_n}} \dfrac{\int_{\Delta_n^{j_1}} \varphi(\xi)d\sigma(\xi) \int_{\Delta_n^{j_2}} \gamma(\xi)d\sigma(\xi)}{(g, h_{n,j_1})(g, h_{n,j_2})(g, h_{n,j})}, & \text{if } \lambda \in \Delta_n^j,\ j \in J_n, \\ 0, & \text{if } \lambda \notin \bigcup_{j\in J_n} \Delta_n^j. \end{cases} \tag{3.1}$$

Here J_n is the set of all j for which $\sigma(\Delta_n^j) > q_n^{-\frac{3}{2}}$.

Proof. We will need a modification of inequality (1.1). Clearly,

$$\begin{aligned} \sum_{j=0}^{q_n-1} \big\|E(\Delta_n^j)g - (g_n, h_{n,j})h_{n,j}\big\| &\le \\ \le \sum_{j=0}^{q_n-1} \big\|E(\Delta_n^j)g_n - (g_n, h_{n,j})h_{n,j}\big\| &+ \sum_{j=0}^{q_n-1} \big\|E(\Delta_n^j)(g - g_n)\big\| \le \\ &\le 2\sqrt{2}q_n^2\sqrt{f(q_n)} + \sqrt{q_n}\|g - g_n\|. \end{aligned}$$

If $\|g - g_n\| = o(q_n^{-\frac{5}{2}})$, $f(n) = o(q_n^{-8})$, then

$$\sum_{j=0}^{q_n-1} \big\|E(\Delta_n^j)g - (g, h_{n,j})h_{n,j}\big\| = o(q_n^{-2}) \tag{3.2}$$

Set $F_n' := \sum_{j\in J_n} (F, h_{n,j})h_{n,j}$.

$$\begin{aligned} \|F_n - F_n'\| &= \Big\| \sum_{j\notin J_n} (F, h_{n,j})h_{n,j} \Big\| \\ &= \Big\| \sum_{j\notin J_n} (F, E(\Delta_n^j)h_{n,j})h_{n,j} + \sum_{j\notin J_n} (F, h_{n,j}')h_{n,j} \Big\| \\ &\le \Big\| \sum_{j\notin J_n} (F, E(\Delta_n^j)h_{n,j})h_{n,j} \Big\| + \Big\| \sum_{j\notin J_n} (F, h_{n,j}')h_{n,j} \Big\| \\ &\le \Big(\sum_{j\notin J_n} \big|(F, E(\Delta_n^j)h_{n,j})\big|^2 \Big)^{\frac{1}{2}} + \|F\| \sum_{j\notin J_n} \|h_{n,j}'\| \\ &< c_1 \left(q_n q_n^{-\frac{3}{2}} + q_n^2 \sqrt{f(q_n)} \right) \\ &< c_2 q_n^{-\frac{1}{2}}. \end{aligned}$$

Here and below c_i $(i = 1, 2, 3, 4)$ are constants that do not depend of n. In the evaluation of the first group of terms we used the fact that

$$
\begin{aligned}
(F, E(\Delta_n^j)h_{n,j}) &= \int_{\Delta_n^j} \varphi(\lambda)\overline{\theta_{n,j}(\lambda)}d\sigma(\lambda) \\
&\le \sup|\varphi(\lambda)| \int_{\Delta_n^j} \overline{\theta_{n,j}(\lambda)}d\sigma(\lambda) \\
&= \sup|\varphi(\lambda)|(g, E(\Delta_n^j)h_{n,j}) \\
&\le \sup|\varphi(\lambda)|(g, h_{n,j}) \\
&\le \sup|\varphi(\lambda)| \left(\|E(\Delta_n^j)\|g + o(q_n^{-2})\right) \\
&\le \sup|\varphi(\lambda)| \left(q_n^{-\frac{3}{2}} + o(q_n^{-2})\right),
\end{aligned}
$$

since $j \notin J_n$[1]. In the evaluation of the second group of terms we used inequality (1.5). Now set

$$F_n'' := \sum_{j\in J_n} \frac{(F, E(\Delta_n^j)g)}{(g, h_{n,j})} h_{n,j},$$

$$\|F_n'' - F_n'\| = \Big\|\Big(\sum_{j\in J_n} \frac{(F, E(\Delta_n^j)g)}{(g, h_{n,j})} - (F, h_{n,j})\Big)h_{n,j}\Big\|.$$

If $j \in J_n$, then

$$
\begin{aligned}
|(g, h_{n,j})| &> \|E(\Delta_n^j)g\| - \|E(\Delta_n^j)g - (g, h_{n,j})h_{n,j}\| > \\
&> \|E(\Delta_n^j)g\| - o(q_n^{-2}) > \frac{1}{2}\|E(\Delta_n^j)g\| > \frac{1}{2}q_n^{-\frac{3}{2}}.
\end{aligned}
$$

Thus

$$\|F_n'' - F_n'\| \le 2q_n^{\frac{3}{2}} \sum_{j\in J_n} \big\|(F, E(\Delta_n^j)g - h_{n,j})h_{n,j}\big\| \le 2q_n^{\frac{3}{2}}\|F\|o(q_n^{-2}) \le c_3 q_n^{-\frac{1}{2}}.$$

Since F_n'' and $F_n' \in H_n$, $\|F_n - F_n''\| < (c_2 + c_3)q_n^{-\frac{1}{2}}$, then $|F_n'' - F_n| < c_4$. Construct a function G_n'' in a similar way. Since $\|F - F_n''\| \to 0$, $\|G - G_n''\| \to 0$ and $|F|$, $|G|$, $|F - F_n''|$ and $|G - G_n''|$ are bounded, then $\|F(x)G(x) - F_n''(x)G_n''(x)\| \to 0$. We have

$$F_n'' \cdot G_n'' = \sum_{j=1}^{q_n} \sum_{\substack{j_1+j_2=j \\ j_1,j_2\in J_n}} \frac{(F, E(\Delta_n^j)g)(G, E(\Delta_n^j)g)}{(g, h_{n,j_1})(g, h_{n,j_2})} h_{n,j},$$

since $h_{n,j_1}h_{n,j_2} = h_{n,j_1+j_2}$. Let

$$\widetilde{R}_n(x) := \sum_{j\in J_n} \sum_{\substack{j_1+j_2=j \\ j_1,j_2\in J_n}} \frac{(F, E(\Delta_n^j)g)(G, E(\Delta_n^j)g)}{(g, h_{n,j_1})(g, h_{n,j_2})} h_{n,j}.$$

$\widetilde{R}_n(x)$ is a projection of $F_n'' \cdot G_n''$ to the subspace $H_n' \subset H_n$ spanned by vectors $h_{n,j}$, $j \in J_n$. Since obviously $H_n' \to L_2(M, \mu)$, then $\|\widetilde{R}_n - F_n'' \cdot G_n''\| \to 0$. Finally, set

$$R_n(x) := \sum_{j\in J_n} \sum_{\substack{j_1+j_2=j \\ j_1,j_2\in J_n}} \frac{(F, E(\Delta_n^j)g)(G, E(\Delta_n^j)g)}{(g, h_{n,j_1})(g, h_{n,j_2})(g, h_{n,j})} E(\Delta_n^j)g.$$

---

[1] $\theta_{n,j}(\lambda)$ is the spectral representation of the function $h_{n,j}(x)$

$$\|R_n(x) - \widetilde{R}_n(x)\| \le \sum_{j\in J_n} \sum_{\substack{j_1+j_2=j\\ j_1,j_2\in J_n}} \frac{|(F, E(\Delta_n^j)g)(G, E(\Delta_n^j)g)|}{|(g,h_{n,j})||(g,h_{n,j_1})||(g,h_{n,j_2})|}\times$$
$$\times \|E(\Delta_n^j)g - (g,h_{n,j})h_{n,j}\|$$
$$\le \Bigg(\sum_{j\in J_n} \|E(\Delta_n^j)g - (g,h_{n,j})h_{n,j}\|^2 \times$$
$$\times \sum_{j\in J_n} \Big| \sum_{\substack{j_1+j_2=j\\ j_1,j_2\in J_n}} \frac{(F, E(\Delta_n^j)g)(G, E(\Delta_n^j)g)}{(g,h_{n,j_1})(g,h_{n,j_2})(g,h_{n,j})} \Big|^2 \Bigg)^{\frac{1}{2}}.$$

Notice that $|(g,h_{n,j})| > \frac{1}{2}\|E(\Delta_n^j)g\|$ for $j\in J_n$, and

$$\sum_{j\in J_n} \Big| \sum_{\substack{j_1+j_2=j\\ j_1,j_2\in J_n}} \frac{(F, E(\Delta_n^j)g)(G, E(\Delta_n^j)g)}{(g,h_{n,j_1})(g,h_{n,j_2})} \Big|^2 \frac{1}{\|E(\Delta_n^j)g\|^2} = \|R_n(x)\|.$$

Using (3.2), we see that $\|R_n(x) - \widetilde{R}_n(x)\| \le 2\|R_n\|\sqrt{o(q_n^2)} \to 0$ and $\|R_n(x) - F(x)G(x)\| \to 0$. Let $\rho_n^{\varphi,\gamma}(\lambda)$ be the spectral representation of $R_n(x)$. Then

$$\rho_n^{\varphi,\gamma}(\lambda) = \begin{cases} \displaystyle\sum_{\substack{j_1+j_2=j\\ j_1,j_2\in J_n}} \frac{\int_{\Delta_n^{j_1}} \varphi(\xi)d\sigma(\xi) \int_{\Delta_n^{j_2}} \gamma(\xi)d\sigma(\xi)}{(g,h_{n,j_1})(g,h_{n,j_2})(g,h_{n,j})}, & \text{if } \lambda\in\Delta_n^j,\ j\in J_n,\\ 0, & \text{if } \lambda\notin\bigcup_{j\in J_n}\Delta_n^j. \end{cases}$$

□

This formula can be done more expressive by factoring out the mean values $\varphi_{n,j}$ and $\gamma_{n,j}$ of functions φ and γ:

$$\varphi_{n,j} := \int_{\Delta_n^{j_1}} \varphi(\lambda)d\sigma(\lambda), \quad \gamma_{n,j} := \int_{\Delta_n^{j_2}} \gamma(\lambda)d\sigma(\lambda),$$

and a 'coefficient of torsion':

$$r_{n,j} := \frac{(g,h_{n,j})}{E(\|\Delta_n^j)g\|}.$$

Note that for $j\in J_n$, $\frac{1}{2} < |r_{n,j}| < 2$. Formula (3.1) goes over the form

$$\rho_n^{\varphi,\gamma}(\lambda) = \begin{cases} \big(\sigma(\Delta_n^j)\big)^{\frac{1}{2}} \displaystyle\sum_{\substack{j_1+j_2=j\\ j_1,j_2\in J_n}} r_{n,j}\frac{\varphi_{n,j_1}\gamma_{n,j_2}}{r_{n,j_1}r_{n,j_2}} \big(\sigma(\Delta_n^{j_1})\big)^{\frac{1}{2}}\big(\sigma(\Delta_n^{j_2})\big)^{\frac{1}{2}}, \text{if } \lambda\in\Delta_n^j,\ j\in J_n,\\ 0,\ \text{if } \lambda\notin\bigcup_{j\in J_n}\Delta_n^j. \end{cases}$$

Remark 3.2. The speed of approximation required in Theorem 3.1 is quite large. The result of Theorem 3.1 is also true for less speed of approximation, since we used quite coarse estimates in the proof. However, since the multiplication formula in the form (3.1) has not found concrete applications yet, we do not give the appropriate adjustments. Our aim is not to obtain the most exact result but to demonstrate the principal opportunity to obtain the multiplication formula through the spectrum and characteristics of approximation.

It can be easily deduced from Theorem 3.1 than for sufficiently large speed of approximation, the values $(g,h_{n,j})$ form a complete system of invariants for an automorphism. The precise formulation of this statement is the following.

Theorem 3.3. *Suppose that automorphisms $T^{(1)}$ and $T^{(2)}$ admit a cyclic approximation by periodic transformations with speed $o(n^{-8})$ and partitions $\xi_n^{(1)}$ and $\xi_n^{(2)}$ each consist of q_n elements. Let $g^{(1)}$ and $g^{(2)}$ be bounded measurable cyclic functions for $U_{T^{(1)}}$ and $U_{T^{(2)}}$, respectively, and $\|g_n^{(1)}-g^{(1)}\| = o(q_n^{-\frac{5}{2}})$, $\|g_n^{(2)}-g^{(2)}\| = o(q_n^{-\frac{5}{2}})$, $(g^{(1)}, h_{n,j}^{(1)}) = (g^{(2)}, h_{n,j}^{(2)})$. Then the automorphisms $T^{(1)}$ and $T^{(2)}$ are metrically isomorphic.*

Proof. First, let us prove that the spectral measures $\sigma^{(1)}$ and $\sigma^{(2)}$ of the functions $g^{(1)}$ and $g^{(2)}$ are equal. It sufficient to show that for each integer k,

$$(U_{T^{(1)}}^k g^{(1)}, g^{(1)}) = (U_{T^{(2)}}^k g^{(2)}, g^{(2)}).$$

Since

$$\|U_{T_n^{(i)}}^k g_n^{(i)} - U_{T^{(i)}}^k g^{(i)}\| \le \|U_{T_n^{(i)}}^k (g_n^{(i)} - g^{(i)})\| + \|(U_{T_n^{(i)}}^k - U_{T^{(i)}}^k) g^{(i)}\| \to 0,$$

then

$$(U_{T^{(i)}}^k g^{(i)}, g^{(i)}) = \lim_{n\to\infty} (U_{T_n^{(i)}}^k g_n^{(i)}, g^{(i)}) \quad (i = 1, 2).$$

But

$$(U_{T_n^{(1)}}^k g_n^{(1)}, g^{(1)}) = \Big(\sum_{j=0}^{q_n-1} \lambda_{n,j}^k (g^{(1)}, h_{n,j}^{(1)}) h_{n,j}^{(1)}, g^{(1)} \Big) =$$

$$= \sum_{j=0}^{q_n-1} \lambda_{n,j}^k |(h_{n,j}^{(1)}, g^{(1)})|^2 = \sum_{j=0}^{q_n-1} \lambda_{n,j}^k |(h_{n,j}^{(2)}, g^{(2)})|^2.$$

Therefore $\sigma^{(1)} = \sigma^{(2)} = \sigma$. Denote by $V^{(1)}$ and $V^{(2)}$ the spectral representations corresponding to vectors $g^{(1)}$ and $g^{(2)}$:

$$V^{(1)}, V^{(2)} \colon L_2(M,\mu) \to L_2(S^1, \sigma).$$

Then $V = V^{(1)}(V^{(2)})^{-1}$ is a unitary operator on $L_2(M,\mu)$. Denote by L the subset of all bounded functions in $L_2(M,\mu)$. The proof will be completed if we show that the operator V is multiplicative, i. e. for any $F, G \in L$

$$V(F \cdot G) = V(F) \cdot V(G). \tag{3.3}$$

It follows from Theorem 3.1 that equation (3.3) holds when $V^{(1)}F$ and $V^{(1)}G$ are bounded functions. In particular, it is true if F and G are finite linear combinations of functions of the form $U_{T^{(1)}}^m g^{(1)}$. Such linear combinations are dense in $L_2(M,\mu)$, since $g^{(1)}$ is a cyclic vector. The following lemma concludes the proof. □

Lemma 3.4. *Let V be a unitary operator on $L_2(M,\mu)$, H is a dense subset in $L_2(M,\mu)$, $H \subset L$, $V(H) \subset L$, and for $F, G \in H$ $V(F \cdot G) = V(F) \cdot V(G)$. Then the operator V is multiplicative.*

Proof. Let $F \in H$, $G \in L$. Let $G_n \in H$ and $\|G_n - G\| \to 0$. Then $V(F \cdot G_n) = V(F) \cdot V(G_n)$ and

$$\|V(F \cdot G_n) - V(F \cdot G)\| = \|F(G_n - G)\| \le \sup|F| \|G_n - G\| \to 0,$$
$$\|V(F) \cdot V(G_n) - V(F) \cdot V(G)\| \le \sup|V(F)| \|G_n - G\| \to 0.$$

Therefore

$$V(F \cdot G) = \lim_{n\to\infty} V(F \cdot G_n) = \lim_{n\to\infty} V(F) \cdot V(G_n) = V(F \cdot G) \tag{3.4}$$

Now let $F, G \in L$ and as before $G_n \in H$ and $\|G_n - G\| \to 0$. As above, $\|V(F \cdot G_n) - V(F \cdot G)\| \to 0$. It follows from (3.4) that $V(F \cdot G_n) = V(F) \cdot F(G_n)$. The sequences $V(F) \cdot V(G_n)$ and $V(G_n)$ converge in $L_2(M, \mu)$, and hence converge in measure. Select a sequence of numbers n_k such that the sequences $V(F) \cdot V(G_{n_k})$ and $V(G_{n_k})$ converge almost everywhere, say for all $x \in B$, where $\mu(M \setminus B) = 0$. For $x \in B$,

$$\lim_{k\to\infty} (V(F))(x) \cdot V(G_{n_k})(x) = (V(F))(x) \lim_{k\to\infty} V(G_{n_k})(x) = (V(F))(x)V(G)(x),$$

i. e. in sense of convergence almost everywhere $\lim_{k\to\infty} V(F \cdot G_{n_k}) = V(F) \cdot V(G)$. But in $L_2(M, \mu)$, $\lim_{k\to\infty} V(F \cdot G_{n_k}) = V(F \cdot G)$. Hence $V(F) \cdot V(G) = V(F \cdot G)$. □