ON LOCAL ENTROPY

by

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The theorem proved in this paper answers the question raised by Lai-Sang Young and F.Ledrappier during the International Symposium on Dynamical Systems in Rio de Janeiro. The problem appeared in connection with their study of interrelations between the measure theoretic entropy, Lyapunov exponents and dimension-like characteristics of smooth dynamical systems. The paper was written in IMPA shortly after the Symposium. We would like to thank the hosts of the Symposium for the invitation to come and for their warm hospitality.

Let X be a compact metric space with distance function d and $f:X\to X$ be a continuous mapping preserving a Borel probability non-atomic measure m. We assume that $h_m(f)$, the entropy of f with respect to m, is finite. Denote as usual for $x,y\in X$ and a positive integer n

$$d_n^f(x,y) = \max_{0\le i\le n-1} d(f^i x, f^i y)$$

and for $r>0$ let $B_n^f(x,r)$ be the d_n^f-ball about x of radius r.

The following theorem may be considered as a local version of the charactsrization of entropy given in [1] and at the same time it is a topological version of the Macmillan-Breiman theorem [2].

<u>Theorem.</u> For m-almost every $x\in X$

$$\text{(a)} \qquad \lim_{\delta\to 0}\ \liminf_{n\to\infty} \frac{-\log m(B_n^f(x,\delta))}{n} =$$

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$$= \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{-\log m(B_n^f(x,\delta))}{n} \overset{\text{def}}{=} h_m(f,x) \;;$$

(b) $h_m(f,x)$ is f-invariant;

(c) $\int_X h_m(f,x)\, dm = h_m(f)$.

Corollary. If f is ergodic with respect to m, then for almost every x $h_m(f,x) = h_m(f)$.

There exists a proof of this statement which is easier than the argument given below for the general case.

Proof of Theorem. Fix $\delta > 0$ and consider a finite measurable partition ξ such that $\operatorname{diam} \xi \overset{\text{def}}{=} \max_{c \in \xi} \operatorname{diam}(c) < \delta$. Let $c_\xi(x)$ be the element of ξ containing x and $c_n^\xi(x)$ be the element of the partition

$$\xi_n = \xi \vee f^{-1}\xi \vee \dots \vee f^{-n+1}\xi$$

containing x. By the Macmillan-Breiman theorem

$$\lim_{n \to \infty} \frac{-\log m(c_n(x))}{n} \overset{\text{def}}{=} m_\xi(x)$$

exists for a.e. x and

$$\int_X m_\xi(x)\, dm = h(f,\xi) \leq h_m(f).$$

Obviously, $B_n^f(x,\delta) \supset c_n^\xi(x)$ so that for every $\delta > 0$ we have

(1) $$\int_X \limsup_{n\to\infty} \frac{-\log m(B_n^f(x,\delta))}{n}\, dm \leq h_m(f) .$$

We proceed now to the estimate from below. Let ξ_e be the decomposition of X into ergodic components of f with respect to m , $p: X \to X/\xi_e = Y$ be the natural projection and $h(y)$, $y \in Y$, be the entropy of the restriction $f|p^{-1}(y)$ with respect to the conditional measure.

Fix a small positive number α , choose $M>1$ such that

$$\int_{h^{-1}([M,\infty))} h(y)\, dm_Y < \alpha$$

and let for $k = 0,1, \ldots , \left[\frac{M}{\alpha}\right] = K$

$$A_k = p^{-1}(h^{-1}([k\alpha, (k+1)\alpha))),$$

$$A_{K+1} = X - \bigcup_{k=0}^{K} A_k , \quad \text{so that} \quad m(A_{K+1}) < \alpha .$$

Denote $a = (A_0, \ldots , A_K, A_{K+1})$. Let $\xi \geq a$ be a finite measurable partition, then for $x \in A_k$, $k = 0,1, \ldots ,K$, $m_\xi(x) \leq (k+1)\alpha$. If, in addition, ξ is sufficiently fine (e.g. its elements have uniformly small diameters), then

(2) $$k\alpha - \frac{\alpha}{100} \leq m_\xi(x) \leq (k+1)\alpha , \quad x \in A_k, \ k \neq K+1.$$

Fix a number $\gamma > 0$ and choose a partition $b \geq a$ such that

$$h_m(f,b) > h_m(f) - \gamma .$$

For every $x \in X$ there exists a ball centered at x of arbitrarily small radius whose boundary has measure 0. Take a finite cover of X by such balls and construct in the usual way the partition into all possible intersections of those balls. Thus, we obtain an arbitrarily fine partition ζ such that $m(\partial\zeta) = 0$, where by definition $\partial\zeta = \bigcup_{c\in\zeta} \partial c$. In particular, we can approximate b by a partition $\eta \leq \zeta$. Therefore, for any positive number q we have constructed a partition η with the following properties

(3) $$h_m(f,\eta) > h_m(f) - \gamma \ ;$$

for every element $c \in \eta$ there exists a set $A_{k(c)}$ such that

(4) $$m(c \cap A_{k(c)}) > (1-q)m(c) \ ;$$

(5) $$m(\partial\eta) = 0 \ .$$

Notice that the number N of elements in η depends only on α and γ but not on q.

For $\delta > 0$ let

$$U_\delta(\eta) = \left\{x \in X : \text{the } \delta\text{-ball about } x \text{ is not contained in } c_\eta(x)\right\}.$$

Since $\bigcap_{\delta>0} U_\delta(\eta) = \partial\eta$, by (5) $m(U_\delta(\eta)) \to 0$ as $\delta \to 0$, so we can choose $\delta > 0$ such that $m(U_{\delta'}(\eta)) < q$ for every $\delta' < \delta$.

Let us denote for $k = 0,1, \ldots ,K$

$$A'_k = \bigcup_{c\in\eta\, :\, k(c) = k} c \ .$$

In other words, A'_k is the union of those elements of η which lie mostly in A_k. Furthermore, denote

$$D = \bigcup_{k=0}^{K} (A_k' - A_k) \ .$$

By (4), $m(D) < q$, hence, by the Birkhoff ergodic theorem and by the Chebyshev inequality applied to the function

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_D(f^i x) \ ,$$

we obtain that for n large enough

$$(6) \qquad m(\{x \in X : \forall n' \geq n \ \sum_{i=0}^{n'-1} \chi_D(f^i x) < 2\sqrt{q}\, n'\}) > 1 - 2\sqrt{q} \ .$$

The same argument applied to $U_\delta(\eta)$ instead of D gives

$$(7) \qquad m(\{x \in X : \forall n' \geq n \ \sum_{i=0}^{n'-1} \chi_{U_\delta(\eta)}(f^i x) < 2\sqrt{q}\, n'\}) > 1 - 2\sqrt{q} \ .$$

We now apply the Macmillan-Breiman theorem to the partitions η and $\xi = \eta \vee a$ and compare the functions m_ξ and m_η . Obviously

$$m_\xi(x) \geq m_\eta(x) \ .$$

On the other hand, by (3)

$$\int m_\eta(x)\, dm = h_m(f,\eta) \geq h_m(f) - \gamma \geq h_m(f,\xi) - \gamma = \int m_\xi(x)\, dm - \gamma .$$

Therefore, by the Chebyshev inequality

$$m(\{x \in X : m_\xi(x) - m_\eta(x) > \sqrt{\gamma}\}) < \sqrt{\gamma} \ .$$

Thus, for n large enough

$$(8) \quad m\left(\left\{x \in X : \frac{\log m(c_n^{\eta}(x)) - \log m(c_n^{\xi}(x))}{n} < 2\sqrt{\gamma}\right\}\right) > 1 - 2\sqrt{\gamma}.$$

Let $F(x) = k\alpha - \frac{\alpha}{50} - 2\sqrt{\gamma}$, if $x \in A_k$, $k = 1,2,\dots,K$, and $F(x) = 0$ if $x \in A_0$. By (2) and (8) we have

$$m\left(\left\{x \in \bigcup_{k=0}^{K} A_k : \frac{-\log m(c_n^{\eta}(x))}{n} < F(x)\right\}\right) < 3\sqrt{\gamma}.$$

Since $\frac{-\log m(c_n^{\eta}(x))}{n} \to m_{\eta}(x)$ a.e., we have

$$m\left(\left\{x \in \bigcup_{k=0}^{K} A_k : m_{\eta}(x) < F(x)\right\}\right) < 3\sqrt{\gamma},$$

so that for n large enough

$$(9) \quad m\left(\bigcup_{k=0}^{K}\left\{x \in A_k : \forall n' \geq n, \frac{-\log m(c_{n'}^{\eta}(x))}{n'} \geq F(x) - \sqrt{\gamma}\right\}\right) \geq m\left(\bigcup_{k=0}^{K} A_k\right) - 4\sqrt{\gamma}.$$

Let E be the set of points satisfying the conditions described in the left-hand parts of (6), (7) and (9) and denote

$$E_k = E \cap A_k .$$

Let $w_{\eta}^{n}(x) = (c_{\eta}(x), c_{\eta}(fx), \dots, c_{\eta}(f^{n-1}x))$ be the (η,n)-name of x. If $y \in B_n^f(x,\delta)$, then for every $0 \leq i \leq n-1$ either $f^i x$ and $f^i y$ belong to the same element of η or $f^i x \in U_{\delta}(\eta)$. Hence, if $x \in E_k$ and $y \in B_n^f(x,\delta)$, then by (7) the Hamming distance between the (η,n)-names of x and y is less than $2\sqrt{q}$. We shall give now an upper estimate for $m(B_n^f(x,\delta))$ for most of the points in E_k. To do this we shall estimate the measure of the set of points y whose (η,n)-names are $2\sqrt{q}$-close to the (η,n)-name of x. For the number L_n of (η,n)-names which are $2\sqrt{q}$-close to the (η,n)-name of $x \in E_k$ we have (see [1], (1.3))

$$\lim_{n \to \infty} \frac{\log L_n}{n} = 2\sqrt{q}\,\log(N-1) - 2\sqrt{q}\,\log(2\sqrt{q}) - (1-2\sqrt{q})\log(1-2\sqrt{q}),$$

where N is the number of elements in η. So, if n is large enough, then

(10) $$L_n \leq \exp(g(q,N)n),$$

where $g(q,N) = 2\sqrt{q}\,\log(N-1) - 2\sqrt{q}\,\log(2\sqrt{q}) - (1-2\sqrt{q})\log(1-2\sqrt{q})+q.$

Since we first choose N and then q, the number g(q,N) can be made arbitrarily small, e.g.

(11) $$g(q,N) < \frac{\alpha}{100} .$$

Fix k, $0 \leq k \leq K$. We want to estimate the measure of points in E_k whose (η,n)-names have an element of η_n of measure greater than $\exp(-k\alpha + \frac{\alpha}{10})n$ in their Hamming $2\sqrt{q}$-neighborhood. Obviously, the total number of such elements does not exceed

$$\exp\left(k\alpha - \frac{\alpha}{10}\right)n ,$$

hence, by (10) and (11), the total number of elements of η_n in their Hamming $2\sqrt{q}$-neighborhood satisfies

(12) $Q_n \leq \exp(k\alpha - \frac{\alpha}{10} + g(q,N))n < \exp(k\alpha - \frac{\alpha}{10} + \frac{\alpha}{100})n$.

Consider those of the Q_n elements of η_n whose intersections with E_k have positive measure. To estimate their total measure S_n we multiply their number estimated by (12) and the upper bound for their measure given by (9):

(13) $S_n \leq \exp\ (k\alpha - \frac{\alpha}{10} + \frac{\alpha}{100} - k\alpha + \frac{\alpha}{50} + 3\sqrt{\gamma})n \leq \exp\ (-\frac{\alpha}{20}n)$,

the last inequality being true if γ is chosen small compared with α.

Thus, we obtain from (13) that

$$m(\{x \in E_k : m(B_n^f(x,\delta)) > \exp\ (-k\alpha + \frac{\alpha}{10})n\}) < \exp\ (-\frac{\alpha}{20}n)$$

for $k = 1,2,\ldots,K$.

By the Borel-Cantelli lemma, for a.e. $x \in E_k$

(14) $$\liminf_{n \to \infty} \frac{-\log m(B_n^f(x,\delta))}{n} \geq k\alpha - \frac{\alpha}{10} .$$

We integrate (14) and get from (6), (7) and (9)

(15) $$\int_X \liminf_{n \to \infty} \frac{-\log m(B_n^f(x,\delta))}{n}\, dm \geq \sum_{k=1}^{K} (k\alpha - \frac{\alpha}{10})\ m(E_k)$$

$$\geq \sum_{k=1}^{K} k\alpha \cdot m(A_k) - \sum_{k=1}^{K} k\alpha \cdot (m(A_k) - m(E_k)) - \frac{\alpha}{10}$$

$$h_m(f) - 2\alpha - \frac{K(K+1)}{2}\alpha(4\sqrt{q} + 4\sqrt{\gamma}) - \frac{\alpha}{10} .$$

Since q and γ were chosen after K and α were, the last expression canbe made arbitrarily close to $h_m(f)$. To achieve that we may have to take δ , which depends on q and γ , very small. Statements (a) and (c) follow immediately from (1) and (15).

Since

$$B_n^f(fx,\delta) \supseteq B_{n+1}^f(x,\delta),$$

it follows from (a) that

$$h_m(f,x) \geqslant h_m(f,fx) \ .$$

The last inequality together with (c) gives (b).

Remark 1. We considered only the case of finite entropy which is the one interesting for applications in smooth dynamical systems. A slightly more complicated argument allows to include transformations with infinite entropy as well.

Remark 2. Our theorem easily implies Theorem 1.1 from [1] as well as its generalization to the non-ergodic case [3], which give the description of measure theoretic entropy in terms of asymptotic capacity of sets of typical orbits.

REFERENCES

1. A.Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Publ.Math.I.H.E.S., v.51(1980), 137-173.
2. P.Billingsley, Ergodic Theory and Information, New York, John Wiley, 1965.
3. A.Katok, Hyperbolicity in smooth dynamical systems, Ecole d'eté de Physique Theorique, Les Houches, 1981,Proc.,to appear.

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