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UNIQUENESS OF LARGE INVARIANT MEASURES FOR $\mathbb{Z}^k$ ACTIONS WITH CARTAN HOMOTOPY DATA

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ABSTRACT. Every C^2 action α of $\mathbb{Z}^k$, $k \geq 2$, on the $(k+1)$-dimensional torus whose elements are homotopic to the corresponding elements of an action α_0 by hyperbolic linear maps has exactly one invariant measure that projects to Lebesgue measure under the semiconjugacy between α and α_0. This measure is absolutely continuous and the semiconjugacy provides a measure-theoretic isomorphism. The semiconjugacy has certain monotonicity properties and preimages of all points are connected. There are many periodic points for α for which the eigenvalues for α and α_0 coincide. We describe some nontrivial examples of actions of this kind.

1. PRELIMINARIES

1.1. **Introduction.** This paper constitutes a direct continuation of [3]. We shall use the terminology and results from [3] with only occasional references.

Let α_0 be a $\mathbb{Z}^k$ Cartan action on $\mathbb{T}^{k+1}$ and let α be a smooth $\mathbb{Z}^k$ action whose elements are homotopic to the corresponding elements of α_0.[1] "Smooth" in our context means C^2 although most of the arguments hold for $C^{1+\varepsilon}$ actions for any positive ε and on the other hand for C^∞ actions various geometric structures we describe are also C^∞. Those include leaves of various stable and unstable foliations, affine structures on those leaves and regularity of the semiconjugacy on leaves of good points.

We will say that such an action α has *Cartan homotopy data* or simply call it *homotopically Cartan.* The action α_0 will be referred to as the *linear model* for α.

Let $h\colon \mathbb{T}^{k+1} \to \mathbb{T}^{k+1}$ be the semiconjugacy between α and α_0, i.e., a unique continuous map $h\colon \mathbb{T}^{k+1} \to \mathbb{T}^{k+1}$ homotopic to identity such that

$$h \circ \alpha = \alpha_0 \circ h \tag{1.1}$$

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[1] In general one should not expect such actions to be actually homotopic in the space of $\mathbb{Z}^k$-*actions*, whether differentiable or continuous. See Section 4 for a discussion of some negative results in this direction

and μ be an ergodic α-invariant measure such that $h_*\mu = \lambda$, the Lebesgue measure.

In this paper we solve Conjecture 1.5 left open in [3] (see Corollary 2.2 below) and obtain further structural information about the semiconjugacy h. Our arguments in a very essential way use results of [3]; in particular, absolute continuity of μ [3, Theorem 1.3], equality of Lyapunov exponents for α and α_0 [3, Theorem 1.7] and smoothness of the semiconjugacy along one-dimensional Lyapunov foliations of α [3, Lemma 4.4].

In Section 3 we study periodic orbits of the action α and derive existence of many such orbits whose eigenvalues are equal to those for the corresponding orbits of α_0. Since periodic orbits form a set of μ-measure zero, results about periodic orbits do not follow directly from those about μ.

In Section 4 we describe families of examples of homotopically Cartan actions that are not topologically conjugate to their linear models. In fact, those actions also cannot be obtained from their linear models via a continuous deformation through C^2 $\mathbb{Z}^k$ actions.

1.2. **Notation.** We shall work on the universal cover of the torus so that unlike [3] we will use the same notations for the objects on the torus or their lifts to the universal cover, unless this leads to confusion, in which case we will indicate the difference.

For a given Weyl chamber C, we shall denote by $\tilde{\mathcal{W}}_C^-$ the corresponding global stable foliation[2] and by $\tilde{\mathcal{W}}_C^+$ the corresponding unstable foliation for α. Let χ_i, $i = 1,\dots,k+1$ be the Lyapunov characteristic exponents for the action α with respect to the measure μ; by [3, Theorem 1.7] they are equal to those of α_0.

For each χ_i there is exactly one Weyl chamber C_i for which $\chi_i < 0$ and $\chi_j > 0$ for all $j \neq i$. The stable foliation $\tilde{\mathcal{W}}_{C_i}^-$ has one-dimensional leaves and coincides with the Lyapunov foliation for the exponent χ_i; to simplify notation slightly we denote this foliation by $\tilde{\mathcal{W}}_i^-$ and the complementary k-dimensional (codimension-one) foliation $\tilde{\mathcal{W}}_{C_i}^+$ by $\tilde{\mathcal{W}}_i^+$. Recall that by definition each leaf of any of those foliations passes through at least one regular point.

DEFINITION 1.1. A set $B \subset \mathbb{R}^{k+1}$ is called a *regular box on the universal cover* if there is a diffeomorphism of B onto the standard cube $\mathbb{D}^{k+1}$ such that for each $i = 1,\dots,k+1$ the preimages of a pair of opposite faces belong to leaves of $\tilde{\mathcal{W}}_i^+$ and the preimage of any face contains a regular point for α.

If B is a regular box on the universal cover and its projection on the torus $\mathbb{T}^{k+1}$ is injective this projection is called a *regular box on the torus*.

We will denote the one-dimensional contracting eigenspace for all elements of α_0 in the Weyl chamber C_i by E_i^- and the k-dimensional expanding subspace for those elements by E_i^+.

[2]See [3, Section 2.2] for the use of word "foliation" in this context

REMARK 1.2. In this paper we will be concerned only with one-dimensional and codimension-one invariant foliations. However, a considerable part of the arguments which appear either in [3] or in the present paper apply to actions with more general homotopy data than Cartan. In order to treat those cases (for example, to prove existence of an absolutely continuous invariant measure) one would have to consider invariant foliations of intermediate dimensions. This topic will be considered in a subsequent paper.

2. MEASURABLE ISOMORPHISM AND UNIQUENESS OF THE ABSOLUTELY CONTINUOUS INVARIANT MEASURE

2.1. Formulation of results and outline of proof.

THEOREM 2.1. *For every $y \in \mathbb{T}^{k+1}$, $h^{-1}(y)$ is the decreasing intersection of regular boxes on the torus.*

Moreover, $h^{-1}(h(y)) = y$ for μ-almost every point $y \in \mathbb{T}^{k+1}$.

The second statement of the theorem immediately implies the following corollary which is the main result of the present paper.

COROLLARY 2.2. *The map h is injective on a set of full measure and thus h is a measure isomorphism between (α, μ) and (α_0, λ). Furthermore, there is only one μ such that $h_* \mu = \lambda$.*

Since every non-identity element of α_0 is a Bernoulli automorphism with respect to λ we have

COROLLARY 2.3. *Every non-identity element of α is a Bernoulli automorphism with respect to μ.*

The uniqueness statement in Corollary 2.2 as well as the fact that λ is the only positive entropy measure for α_0 [7, 2] justifies calling μ *the large invariant measure* for α. Isomorphism rigidity for Cartan actions [4] implies a similar fact for homotopically Cartan actions.

COROLLARY 2.4. *Let $\alpha^{(1)}$ and $\alpha^{(2)}$ be two homotopically Cartan actions. They are measurably isomorphic with respect to their large invariant measures if and only if their linear models ${\alpha_0}^{(1)}$ and ${\alpha_0}^{(2)}$ are algebraically isomorphic.*

Another immediate corollary of Theorem 2.1 is of a purely topological nature.

COROLLARY 2.5. *The preimage $h^{-1}(y)$ of any point $y \in \mathbb{T}^{k+1}$ is connected.*

The scheme of proof of Theorem 2.1 is in short as follows.

Step 1. The leaves of the Lyapunov foliation $\tilde{\mathcal{W}}_i^-$ have a smooth α-invariant affine structure and for almost every leaf the semiconjugacy maps the leaf affinely onto a line in $\mathbb{R}^{k+1}$.

Step 2. The leaves of the codimension-one foliation $\tilde{\mathcal{W}}_i^+$ possess a smooth α-invariant affine structure and for almost every leaf the semiconjugacy maps the leaf affinely onto a codimension-one linear subspace in $\mathbb{R}^{k+1}$. Such a leaf is a smoothly embedded codimension-one submanifold of $\mathbb{R}^{k+1}$. Call such leaves *good leaves* and any regular point on a good leaf a *good point*.

Step 3. The semiconjugacy preserves the closed half-spaces into which any good leaf divides $\mathbb{R}^{k+1}$.

Step 4. The order in the set of good leaves of $\tilde{\mathscr{W}}_i^+$ is given by the order of their unique intersection points with almost any leaf of the complementary Lyapunov foliation $\tilde{\mathscr{W}}_i^-$.

Step 5. Regular boxes are constructed as intersections of "slices" between two good leaves of $\tilde{\mathscr{W}}_i^+$ for $i = 1, \dots, k+1$.

Step 6. Almost every point belongs to an arbitrarily small regular box.

Before proceeding to a detailed discussion let us emphasize that in the proof arguments of three kinds are used:

- *Geometric:* Existence of invariant densities and affine structures; those are fairly general and are based on contraction estimates for various elements of the action and on commutativity. Properly modified versions of those arguments extend to considerably more general situations than the homotopically Cartan actions considered presently.
- *Ergodic:* Smoothness of conditional measures and rigidity of affine structures almost everywhere. Those arguments use recurrence essentially. One-dimensionality of the Lyapunov foliations is not essential but uniform C^0 bounds along certain directions are.
- *Topological:* Separation of the space into invariant blocks by codimension-one manifolds. Here existence of at least one codimension-one invariant foliation is essential but one can go beyond the homotopically Cartan case using Denjoy-type arguments instead of relying on the full set of codimension-one foliations as we do.

2.2. **Semiconjugacy along the leaves of $\tilde{\mathscr{W}}_i^-$.** Since this has already been done in [3] we just quickly summarize this step.

Every leaf $\mathscr{L}$ of the Lyapunov foliation $\tilde{\mathscr{W}}_i^-$ is the image of a real line under a smooth immersion and it contains (in general many) *local leaves* around regular points on the leaf. Each local leaf is the image of an interval under a smooth embedding with bounded inverse. Notice that we do not know *a priori* whether the global leaf (on the universal cover) is the image of $\mathbb{R}$ under a diffeomorphic embedding.

Moreover, any global leaf has a uniquely defined α-invariant smooth affine structure which agrees on any local leaf inside the global leaf.[3] The semiconjugacy h maps the leaf $\mathscr{L}$ onto a line L parallel to the corresponding eigenspace of α_0 (this relies on one-dimensionality). The crucial point is that *for almost every leaf* the semiconjugacy intertwines the affine structures on $\mathscr{L}$ and L.

[3] The regularity of the affine structure on each leaf of $\tilde{\mathscr{W}}_i^-$ and hence of the restriction of h to almost every leaf is essentially the same as that of α; in particular it is C^∞ for a C^∞ action α.

LEMMA 2.6. [3, Lemma 4.4] *For almost every point x and every Lyapunov foliation $\tilde{\mathcal{W}}_i^-$ the semiconjugacy h is a $C^{1+\epsilon}$ affine diffeomorphism from $\tilde{\mathcal{W}}_i^-(x)$ onto the line $h(x)+E_i^-$.*

Those leaves that satisfy the assertion of the lemma are closed diffeomorphically embedded submanifolds of $\mathbb{R}^{k+1}$.

We point out that in the proof of this lemma recurrence arguments are used in an essential fashion. We do not prove that *every* leaf which contains regular points has this property. At this step the existence of a direction in the action (or rather in its suspension) along which α_0 acts by isometries is crucial.

2.3. **Conditional measures and invariant affine structures on codimension-one foliations.** Next we consider leaves of the codimension-one foliation $\tilde{\mathcal{W}}_i^+$. Similarly to the previous step any such leaf is the image of an $\mathbb{R}^k$ under a smooth immersion (but not necessarily of a diffeomorphic embedding) and it contains *local leaves* around regular points on the leaf. Each local leaf is the image of a ball under a smooth embedding with bounded inverse.

By [3, Lemma 2.2] the semiconjugacy h maps each leaf of $\tilde{\mathcal{W}}_i^+(x)$ into a codimension-one hyperplane parallel to the expanding subspace E_i^+ for all elements of α_0 in the Weyl chamber C_i. We do not know *a priori* whether the semiconjugacy maps an individual leaf to this hyperplane injectively or surjectively, unlike the case of one-dimensional foliations where surjectivity always holds.

PROPOSITION 2.7. *On each leaf of $\tilde{\mathcal{W}}_i^+$ there is a unique smooth α-invariant affine structure together with a k-frame such that for any regular point x and $j \neq i$ the one-dimensional leaf $\tilde{\mathcal{W}}_j^-(x)$ is a coordinate line in $\tilde{\mathcal{W}}_i^+(x)$ and for any regular $z \in \tilde{\mathcal{W}}_i^+(x)$ the affine structure on $\tilde{\mathcal{W}}_j^-(z)$ coincides with the restriction of the affine structure on $\tilde{\mathcal{W}}_i^+(x)$.*

Proof. We use the theory of normal forms for extensions of measure-preserving transformations by smooth contractions as outlined in [2, Section 6.2]. Note that in the absence of resonances between Lyapunov exponents the nonstationary normal form is affine. One can pick an element $\mathbf{m} \in -C_i$ such that there are no resonances between the negative Lyapunov exponents $\chi_j(\mathbf{m})$, $j \neq i$. In fact, one can make those exponents almost equal (and even equal if one passes to the suspension; this will be used in the proof of the next lemma). Hence by [2, Theorem 6.2] an invariant affine structure exists for $\alpha(\mathbf{m})$ and by [2, Theorem 6.3] it is unique and is invariant for the whole action α.

Now for a given $j \neq i$ one can find $\mathbf{m} \in -C_i$ for which $\chi_j(\mathbf{m}) < \chi_k(\mathbf{m})$ for all $k \neq j$. Hence the leaves of $\tilde{\mathcal{W}}_j^-$ inside a leaf of $\tilde{\mathcal{W}}_i^+$ are the fastest contracting manifolds which are uniquely defined, and for affine maps they must be lines. □

LEMMA 2.8. *The density of the conditional measure on almost every leaf of $\tilde{\mathcal{W}}_i^+(x)$ is constant with respect to the affine structure and is hence positive and smooth.*

Proof. The only essential part is positivity because then the standard telescopic ratio of Jacobians gives a unique formula for the ratio of densities; see e.g., the formula for $\rho_x(z)$ in the proof of Lemma 3.2 in [3]. This formula holds for arbitrary stable manifolds, not only for one-dimensional ones. This is stated in [8, Section 6.1] as displayed but unnumbered formula on p. 533. There are minor deficiencies in the proof of positivity on pp. 533–534 of [8] which the second author has been able to correct (unpublished). Thus the above reference is valid.

We outline here in a somewhat sketchy way an alternative more geometric argument for our special case which may be of independent interest.

Consider the suspension of the action α (which we will still denote by α) and take a typical point x. Let $\mathbf{t} \in \mathbb{R}^k$ be an element in the center of the Weyl chamber $-C_i$, i.e., an element for which all negative exponents are equal. Consider the action of the suspension of $\alpha(n\mathbf{t})$ for positive integers n. The Lyapunov exponents $\chi_j(\mathbf{t})$ for $j \neq i$ are negative and equal and by [3, Lemma 4.2] the restriction of $\alpha(n\mathbf{t})$ to the leaf of x corresponding to $\tilde{\mathcal{W}}_i^+$ are conformal contractions with exponentially decreasing coefficients with respect to a measurable Riemannian metric. Since among the points $\alpha(n\mathbf{t})(x)$ there are infinitely many Lebesgue density points of the set where the density is positive and which also belong to a fixed Pesin set, the proportion of the conditional measure in the image will become arbitrarily high. But since this proportion is preserved it must be equal to one to begin with. □

REMARK 2.9. The above argument strongly relies on properties of homotopically Cartan actions; it can be extended to certain more general situations if one replaces the standard Lebesgue density point argument by a reference to stronger results in geometric measure theory.

PROPOSITION 2.10. *For almost every regular point z the restriction of the semiconjugacy h to the leaf $\tilde{\mathcal{W}}_i^+(z)$ is an affine bijection between $\tilde{\mathcal{W}}_i^+(z)$ and the hyperplane $h(z)+E_i^+$.*

Proof. Take z for which almost every point of the leaf $\tilde{\mathcal{W}}_i^+(z)$ with respect to the k-dimensional volume is regular. By Lemma 2.8 the set of such points is of full μ measure. Thus there is a dense subset of $\tilde{\mathcal{W}}_i^+(z)$ where leaves of $\tilde{\mathcal{W}}_j^-$ for all $j \neq i$ are defined. By Proposition 2.7 any such manifold is a part of a corresponding line and its affine parameterization agrees with the one coming from the affine structure in $\tilde{\mathcal{W}}_i^+(z)$. But we already know that the semiconjugacy on any leaf of $\tilde{\mathcal{W}}_j^-$ is affine. Furthermore for each regular $y \in \tilde{\mathcal{W}}_i^+(z)$ the manifold $\tilde{\mathcal{W}}_j^-(y)$ cannot be just an interval but must be the whole line in the affine structure. Thus we know that h restricted to $\tilde{\mathcal{W}}_i^+(z)$ is affine on a dense set of lines parallel to each coordinate direction. Hence it is affine. □

LEMMA 2.11. *For any point z satisfying the assertion of Proposition 2.10, the manifold $\tilde{\mathcal{W}}_i^+(z)$ is diffeomorphically embedded into $\mathbb{R}^{k+1}$.*

Proof. Proposition 2.10 implies this statement for any compact part of $\tilde{\mathscr{W}}_i^+(z)$. But since h is a bounded distance away from the identity for any sequence of points on $h(z)+E_i^+$ that goes to infinity, the preimages go to infinity too. □

2.4. **Preservation of the order of codimension-one leaves by the semiconjugacy.** Recall that we call points satisfying the assertion of Proposition 2.10 good points. Given a good point x, we have that the manifold $\tilde{\mathscr{W}}_i^+(x)$ is mapped onto $h(x)+E_i^+$ and $\tilde{\mathscr{W}}_i^-(x)$ is mapped diffeomorphically onto $h(x)+E_i^-$. Let us fix an ordering on the line E_i^- and an ordering on each $\tilde{\mathscr{W}}_i^-(x)$ in such a way that h preserves the order. As each leaf of $\tilde{\mathscr{W}}_i^+$ divides $\mathbb{R}^{k+1}$ into two connected components and is homeomorphic to $\mathbb{R}^k$ we can put a transverse orientation on each leaf of $\tilde{\mathscr{W}}_i^+$ that is coherent with the orientation given to $\tilde{\mathscr{W}}_i^-(x)$.

DEFINITION 2.12. Given a leaf L of $\tilde{\mathscr{W}}_i^+$ we say that a point $y\in\mathbb{R}^{k+1}$ is *below* L if $y\notin L$ and there is a curve $\gamma(t)$, $t\in[0,1]$ such that $\gamma(0)=y$, $\gamma(1)\in L$, $\gamma[0,1)$ does not intersect L and $\dot\gamma(1)$ is positive with respect to the transverse orientation given to L. Similarly we define y being *above* L.

Observe that any point is either on the leaf L or above it or below it. Given a leaf L of $\tilde{\mathscr{W}}_i^+$ we define L^- as the set of points below L and L^+ the above ones. Observe that $\alpha(\mathbf{m})(L^-)=\alpha(\mathbf{m})(L)^-$ for every $\mathbf{m}\in\mathbb{Z}^k$ and $L^-+\mathbf{k}=(L+\mathbf{k})^-$ for every $\mathbf{k}\in\mathbb{Z}^{k+1}$. For two leaves L_1 and L_2 of $\tilde{\mathscr{W}}_i^+$ we say that $L_2>L_1$ if every point in L_1 is below L_2.

LEMMA 2.13. *Let L_1 and L_2 be two different good leaves of $\tilde{\mathscr{W}}_i^+$ then $h(L_1)\neq h(L_2)$ Moreover, if $L_2>L_1$ then $h(L_2)>h(L_1)$.*

Proof. Take a good point $y\in L_1$. Then $\tilde{\mathscr{W}}_i^-(y)\cap L_2=z$ by Lemma 2.6 and Proposition 2.10, and thus z and y lie on the same leaf of the Lyapunov foliation $\tilde{\mathscr{W}}_i^-$ and z lies above y. Since the restriction of h to a leaf of $\tilde{\mathscr{W}}_i^-$ is an orientation-preserving diffeomorphism the lemma is proved. □

LEMMA 2.14. *Let L_1 and L_2 be two different good leaves of $\tilde{\mathscr{W}}_i^+$. Then there is a constant $C>0$ such that for every $\mathbf{m}\in\mathbb{Z}^k$ and $n<0$,*

$$d(\alpha(n\mathbf{m})(L_1),\alpha(n\mathbf{m})(L_2))\geq Ce^{n\chi_i(\mathbf{m})}-C.$$

Proof. This follows immediately from the properties of the semiconjugacy and Lemma 2.13. □

LEMMA 2.15. *If $L_2>L_1$ and $z\in L_1^-$ then $d(z,L_2)>d(L_1,L_2)$.*

Proof. This is just a consequence of the fact that the distance between two closed sets is attained either on the boundary or not at all. □

LEMMA 2.16. *Let $L_2>L_1$ be two good leaves of $\tilde{\mathscr{W}}_i^+$. Take $y_2\in L_2$ and $y_1\in L_1^-$ then for every $\mathbf{m}\in\mathbb{Z}^k$ and $n<0$*

$$d(\alpha(n\mathbf{m})(y_1),\alpha(n\mathbf{m})(y_2))\geq Ce^{n\chi_i(\mathbf{m})}-C \tag{2.1}$$

for $n<0$.

Proof. This is an immediate consequence of Lemmas 2.14 and 2.15. □

LEMMA 2.17. *If y is below L then $h(y)$ is not above $h(L)$.*

Proof. Assume by contradiction that there are y and L, with y below L but $h(y)$ above $h(L)$. Take a good leaf L' above L such that $h(L')$ is still below $h(y)$. Such a leaf exists by Proposition 2.10. Take a path γ from y to L as in Definition 2.12. Then $h(\gamma)$ is a curve (continuous image of the interval [0,1]) one of whose endpoints is $h(y)$ and the other endpoint is in $h(L)$. As $h(L')$ is below $h(y)$ but above $h(L)$, hence between them, $h(L')$ must intersect $h(\gamma)$. Take a point $\hat{y} \in \gamma$ such that $h(\hat{y}) \in h(L')$. On the other hand, by Proposition 2.10 there is another point $y' \in L'$ such that $h(y') = h(\hat{y})$. Since $\hat{y} \in L^-$ we have by Lemma 2.16 that for $\mathbf{m} \in -C_i$ the distance between $\alpha(n\mathbf{m})(y')$ and $\alpha(n\mathbf{m})(\hat{y})$ should grow exponentially fast which contradicts the fact that $h(x) = h(y)$ if and only if distances along the orbit stay bounded in the universal cover. □

2.5. **Construction of regular boxes.** Given intervals $[a_i^-, a_i^+] \subset \tilde{\mathscr{W}}_i^-(x)$ whose endpoints $a_i^- < a_i^+$ are good points for all $i = 1, \dots k+1$, we define the box

$$[a_1^-, a_1^+] \times \cdots \times [a_{k+1}^-, a_{k+1}^+]$$

as the set of points lying on or below $\tilde{\mathscr{W}}_i^+(a_i^+)$ and on or above $\tilde{\mathscr{W}}_i^+(a_i^-)$ for every $i \in [1, \dots, k+1]$.

LEMMA 2.18.

$$h\left([a_1^-, a_1^+] \times \cdots \times [a_{k+1}^-, a_{k+1}^+]\right) = [h(a_1^-), h(a_1^+)] \times \cdots \times [h(a_{k+1}^-), h(a_{k+1}^+)].$$

The semiconjugacy maps the boundary of the box

$$[a_1^-, a_1^+] \times \cdots \times [a_{k+1}^-, a_{k+1}^+]$$

diffeomorphically onto the boundary of the corresponding box for the linear action α_0, preserving the standard decomposition into faces of each dimension.

Proof. The first claim is an immediate consequence of Lemma 2.17 and the second one of Proposition 2.10. □

2.6. **Conclusion of the proof of Theorem 2.1.** Take a point y and for each i let

$$y_i^+ := \inf\{z \in \tilde{\mathscr{W}}_i^-(x) : \ y \text{ is below } \tilde{\mathscr{W}}_i^+(z)\}$$

and

$$y_i^- := \sup\{z \in \tilde{\mathscr{W}}_i^-(x) : \ y \text{ is above } \tilde{\mathscr{W}}_i^+(z)\}.$$

Observe that $y_i^+ = y_i^- = y_i$ because good points are dense in $\tilde{\mathscr{W}}_i^-(x)$.

For $i = 1, \dots, k+1$, take some sequences of good points

$$a_{i,n}^- < y_i < a_{i,n}^+$$

with $a_{i,n}^-$ growing to y_i and $a_{i,n}^+$ decreasing to y_i. Thus the box

$$[a_{1,n+1}^-, a_{1,n+1}^+] \times \cdots \times [a_{k+1,n+1}^-, a_{k+1,n+1}^+]$$

fits inside the box

$$[a_{1,n}^-, a_{1,n}^+] \times \cdots \times [a_{k+1,n}^-, a_{k+1,n}^+]$$

Now we will show that that the decreasing intersection of these boxes is exactly $h^{-1}(h(y))$ and moreover that, $h^{-1}(h(y)) = y$ for μ-a.e. y.

We know that h is an orientation-preserving diffeomorphism when restricted to any leaf of any Lyapunov foliation so that $h(a^-_{i,n})$ grows to $h(y_i)$ and $h(a^+_{i,n})$ decreases to $h(y_i)$ as $n \to \infty$ for every i. By Lemma 2.18 the image of the intersection of

$$[a^-_{1,n}, a^+_{1,n}] \times \cdots \times [a^-_{k+1,n}, a^+_{k+1,n}]$$

for $n = 1, 2, \ldots$ is exactly $h(y)$. Take now a point $z \in h^{-1}(h(y))$ and assume by contradiction that it is not in

$$[a^-_{1,n}, a^+_{1,n}] \times \cdots \times [a^-_{k+1,n}, a^+_{k+1,n}]$$

for some n. Then z should be above $\tilde{\mathscr{W}}_i^+(a^+_{i,n})$ or below $\tilde{\mathscr{W}}_i^+(a^-_{i,n})$ for some $1 \leq i \leq k+1$. If it is below $\tilde{\mathscr{W}}_i^+(a^-_{i,n})$ then, by Lemma 2.17, $h(z) = h(y)$ is not above $h(a^-_{i,n}) + E^+_i$. But then it should be below $h(a^-_{i,n+1}) + E^+_i$ contradicting that y is above $\tilde{\mathscr{W}}_i^+(a^-_{i,n+1})$ again by Lemma 2.17. Thus we get that z is in the intersection of the boxes. As an immediate consequence we conclude that $h^{-1}(h(y))$ is connected.

Take now y from a given Pesin set Λ and assume that y is approached from both sides on each $\tilde{\mathscr{W}}_i^-(y)$ by points of Λ. This is true for μ−a.e. $y \in \Lambda$.

Let us show that $h^{-1}(h(y)) = y$, that is, that the intersection of the boxes is exactly y. To this end, we shall take the sequences $a^{\pm}_{i,n}$ in Λ. Then $\tilde{W}_i^+(a^{\pm}_{i,n}) \to \tilde{W}_i^+(y)$ by the continuity of the foliations on the Pesin set. Thus for big enough n we can build a small box B_n whose boundary is inside the local manifolds $\tilde{W}_i^+(a^{\pm}_{i,n})$ and decreasing towards y. A priori these boxes may not be the same boxes we have already defined. If $h^{-1}(h(y)) \neq y$ then for some big enough n there should be a point in $h^{-1}(h(y))$ that is not in B_n, but then as $h^{-1}(h(y))$ is connected there should be a point in $h^{-1}(h(y))$ that is on the boundary of B_n, contradicting Lemma 2.13. Taking Pesin sets of arbitrarily large measure we obtain a full-measure set. This finishes the proof of Theorem 2.1. □

3. PERIODIC ORBITS OF ACTIONS WITH CARTAN HOMOTOPY DATA

3.1. **Proper periodic orbits.** For the moment we speak about actions on the torus, not on the universal cover. Any periodic (finite) orbit of α is mapped by the semiconjugacy h onto a periodic orbit of the linear action α_0. We will call the stationary subgroup of such an orbit its *period*. Periodic orbits of α_0 are exactly those which consist of points with rational coordinates; the set $\mathbb{Q}^{m+1}/\mathbb{Z}^{m+1}$ of all such points is evidently α_0-invariant.

In this section we will study those periodic points p of α for which $h^{-1}(h(p)) = p$ and for which the eigenvalues for all elements of the stationary subgroup coincide with the eigenvalues of the corresponding elements of α_0 at $h(p)$. Let us call such periodic points and their orbits *proper*. Notice that the existence of a proper periodic point is not *a priori* obvious; neither does it follow directly from the results about behavior and properties of the absolutely continuous invariant measure. However, some of the arguments used in the previous section to establish those properties can be properly modified and supplemented to establish abundance of proper periodic points. Notice that all additional arguments we introduce in this section are geometric in the sense described at the end of Section 2.1.

It is not known if the preimage of any α_0-periodic point contains an α-periodic point. Of course, there is always a periodic point for any element of the action in this preimage and if such a point is hyperbolic then it will be periodic for the action, but the period may change.

THEOREM 3.1. *Proper periodic points are dense in the support of the large α-invariant measure μ.*

REMARK 3.2. We can also prove that the number of proper periodic orbits of a given period in certain directions (such as those described in the next paragraph) grows exponentially with the period and give lower bounds for the speed of growth. Since so far this result looks far from optimal we do not present it here.

3.2. **Proof of Theorem 3.1.** Let G be the set of all $\mathbf{m} \in \mathbb{Z}^k$ such that exactly one among the Lyapunov exponents $\chi_i(\mathbf{m})$, $i = 1, \dots, k+1$ is negative and the rest are positive, different and with 2:1 bunching, i.e., the minimal positive exponent is strictly greater than one half of the maximal one.

Fix an $\mathbf{m} \in G$ and take a good regular point x for α with a close return in the $\mathbf{m}$ direction and construct a periodic point p for $\mathbf{m}$ by shadowing close to x, as in [6, Lemma S.4.10]. Then p lies inside a small regular box and the Lyapunov exponents at p for $\alpha(\mathbf{m})$ are close to the corresponding exponents $\chi_i(\mathbf{m})$. In particular there is still one negative exponent, and the rest are positive, different and with 2:1 bunching. Moreover, by the 2:1 bunching we have that $\alpha(\mathbf{m})$ restricted to the unstable manifold of p leaves invariant a unique affine structure that is close to the one at x. Observe that, *a priori*, the affine structure at p, unlike that at x, is only invariant by $\alpha(\mathbf{m})$ and not by the rest of the action. In fact an essential part of the proof is to show that it is invariant by other elements of the action.

The one-dimensional local stable manifold of p for $\alpha(\mathbf{m})$ is close to the one of a regular point and so it should be big enough such that both branches cut the boundary of the small regular box it is inside. Moreover the local unstable manifold of p is also close to a regular one so that its intersection with the boundary of the box bounds a disc in the local unstable manifold with p in its interior.

Denote the global stable and unstable manifolds of p for $\alpha(\mathbf{m})$ by S and U correspondingly.

LEMMA 3.3. *The point p is periodic for the whole action α with the same period as that of $h(p)$ for α_0. Moreover, h maps S and U onto the stable and unstable manifolds of $h(p)$ respectively.*

Proof. Now we return to the arguments on the universal cover. Observe first that as each branch of S cuts a good leaf it must be mapped onto a branch of the stable manifold of $h(p)$. Moreover, if both branches were mapped into the same branch by h that would imply that both branches cut the same good leaf which contradicts injectivity along good leaves. Hence S is mapped onto the stable manifold of $h(p)$. Take any $\mathbf{n} \in \mathbb{Z}^k$ such that $\alpha_0(\mathbf{n})(h(p)) = h(p)$. If $\alpha(\mathbf{n})(p) \neq p$ then the stable manifolds of p of $\alpha(\mathbf{n})(p)$ will cut the same good leaf which contradicts injectivity of the semiconjugacy on good leaves. To see that $h(U)$ is the whole unstable manifold of $h(p)$ it is enough to see that it contains an open set around $h(p)$. If we take the good leaf that is in the" top" of the small box surrounding p, then successive images of it intersected with the box are mapped onto uniform-size disks centered on the stable manifold of $h(p)$ and converge to the local unstable manifold of p. Hence the image of the limit, which is the local unstable manifold of p, is sent onto the a nontrivial disk centered at $h(p)$. Using invariance, we get that U is mapped onto the global unstable manifold of $h(p)$. □

Thus the whole action α preserves the stable and the unstable manifolds of p. Using the following standard fact we get that the Lyapunov exponents in the stable direction are proportional to the corresponding ones of α_0.

LEMMA 3.4. *Let $\rho_*, \rho\colon \mathbb{Z}^N \to \mathbb{R}^+$ be linear actions on the line, and assume that there is a continuous nonconstant map $h\colon \mathbb{R}^+ \to \mathbb{R}^+$, that extends continuously to* 0*, such that $h \circ \rho = \rho_* \circ h$. Assume also that the image of ρ_* is dense in $\mathbb{R}^+$. Then there are $t > 0$ and $c \in \mathbb{R}^+$ such that $h(x) = c|x|^t$ and $\rho_* = \rho^t$.*

Take now a good complementary leaf L passing near x. L is mapped by h affinely and surjectively onto a hyperplane at the unique intersection point of L with S, which we will denote by q. Since h is affine on L there are lines in this structure passing through q which are mapped by h onto Lyapunov lines for α_0. Moreover, all the spaces generated by those lines are close to the ones of p because the invariant affine structure at p is close to the one of x and hence to that on the leaf L. Order those lines in decreasing order of values of the Lyapunov exponents for $\mathbf{m}$, $\chi_{k+1}(\mathbf{m}) < 0 < \chi_k(\mathbf{m}) < \cdots < \chi_1(\mathbf{m})$. This order is the same for α_0 and at p by construction. We will denote the corresponding Lyapunov exponents at p by $\chi_{i,p}$, $i = 1, \dots, k+1$. Thus we already know that there is a $t_{k+1} > 0$ such that $\chi_{k+1}(\mathbf{n}) = t_{k+1}\chi_{k+1,p}(\mathbf{n})$, for every $\mathbf{n} \in \mathbb{Z}^k$.

The fastest line at q corresponding to the exponent χ_1 is close to the fastest line at p. So, applying positive iterates of $\alpha(\mathbf{m})$ we get that the images are of uniform size and converge to the fastest line at p by the standard strong invariant manifold theorem. By continuity the limit is mapped by h to the fastest line at $h(p)$ surjectively. By uniqueness, the fastest invariant manifold for p should be invariant by the whole action, hence, by Lemma 3.4 the corresponding exponents χ_1 and $\chi_{1,p}$ are proportional.

More generally, given the first j Lyapunov exponents, take the j-dimensional affine space at q, $\tilde{\mathscr{P}}_j(q)$, corresponding to these Lyapunov exponents. It should be close to the corresponding one at p and applying positive iterates of $\alpha(m)$ we get that its images are of uniform size and converge to $\tilde{\mathscr{P}}_j(p)$ again by the standard strong invariant manifold theorem. Denoting the corresponding spaces for α_0 without the tilde, as $h(\tilde{\mathscr{P}}_j(q)) = \mathscr{P}_j(h(q))$, by continuity, $\tilde{\mathscr{P}}_j(p)$ is mapped by h to $\mathscr{P}_j(h(p))$ surjectively. Again by uniqueness, $\tilde{\mathscr{P}}_j(p)$ should be invariant by the whole action.

Now we prove by induction that the one-dimensional $\alpha(\mathbf{m})$-invariant manifolds corresponding to different eigenvalues at p are invariant for the whole action α and the corresponding Lyapunov exponents are proportional with positive coefficients of proportionality.

This has already been done for the fastest manifold. Assume that this has been proved for the first $j-1$ Lyapunov exponents and let argue for the jth.

Take an element of the action $\mathbf{m}_j$ such that $\chi_i(\mathbf{m}_j) < 0$ for $i = 1,\dots,j-1$ and $\chi_j(\mathbf{m}_j) > 0$. Then, by the induction hypothesis (the Lyapunov exponents χ_i and $\chi_{i,p}$ are proportional for $i = 1,\dots,j-1$) we should have that $\chi_{i,p}(\mathbf{m}_j) < 0$ for $i = 1,\dots,j-1$.

If $\chi_{j,p}(\mathbf{m}_j) < 0$, then the whole $\tilde{\mathscr{P}}_j(p)$ would be the stable manifold of p for $\alpha(\mathbf{m}_j)$ restricted to $\tilde{\mathscr{P}}_j(p)$ and hence its image by h would be in the stable manifold of $\alpha_0(\mathbf{m}_j)$ restricted to $\mathscr{P}_j(h(p))$ whose dimension is less than $\mathscr{P}_j(h(p))$, contradicting surjectivity of $h\colon \tilde{\mathscr{P}}_j(p) \to \mathscr{P}_j(h(p))$. Hence $\chi_{j,p}(\mathbf{m}_j) \geq 0$.

If $\chi_{j,p}(\mathbf{m}_j) = 0$, we can take another element $\mathbf{m}'_j$ of the form $l\mathbf{m}_j - \mathbf{m}$, $l > 0$ such that the corresponding Lyapunov exponents for α_0 remain the same sign but $\chi_{j,p}(\mathbf{m}'_j) < 0$. Hence $\chi_{j,p}(\mathbf{m}_j) > 0$ and then there is a one-dimensional unstable manifold corresponding to this exponent inside $\tilde{\mathscr{P}}_j(p)$, that should be invariant by the whole action and hence by Lemma 3.4 the jth Lyapunov exponent is proportional to χ_j.

Now we will prove that the Lyapunov exponents coincide. Let us first show that $\chi_{1,p} = \chi_1$. Since we already know that the Lyapunov exponents are proportional it is enough to prove that $\chi_{1,p}(\mathbf{m}) = \chi_1(\mathbf{m})$. Recall that h is a diffeomorphism along good leaves and hence it is a diffeomorphism at q along the good leaf. Take an affine segment γ_l through q of length ε_l in the fastest direction of $\alpha(\mathbf{m})$. Then $h(\gamma_l)$ is a segment at $h(q)$ in the fastest direction of $\alpha_0(\mathbf{m})$ and

$$\frac{\varepsilon_l}{C_q} \leq |h(\gamma_l)| \leq C_q \varepsilon_l.$$

Since q lies on the stable manifold of p and their unstable manifolds exponentially converge under positive iterates of $\alpha(\mathbf{m})$ we have

$$|\alpha(l\mathbf{m})(\gamma_l)| \sim e^{l\chi_{1,p}(\mathbf{m})}|\gamma_l|.$$

On the other hand

$$|\alpha_0(l\mathbf{m})(h(\gamma_l))| = e^{l\chi_1(\mathbf{m})}|h(\gamma_l)|.$$

If we take $\varepsilon_l = \frac{e^{-\chi_{1,p}(l\mathbf{m})}}{l}$ then $|\alpha(l\mathbf{m})(\gamma_l)| \sim 1/l$ and

$$|\alpha_0(l\mathbf{m})(h(\gamma_l))| \sim \frac{e^{l(\chi_1(\mathbf{m}) - \chi_{1,p}(m))}}{l}.$$

So finally, as $\alpha_0(l\mathbf{m})(h(\gamma_l)) = h(\alpha(l\mathbf{m})(\gamma_l))$, we have that $\chi_1(\mathbf{m}) \le \chi_{1,p}(\mathbf{m})$. On the other hand, if we take $\varepsilon_l = \frac{e^{-\chi_1(l\mathbf{m})}}{l}$ then $|\alpha_0(l\mathbf{m})(h(\gamma_l))| \sim 1/l$ and

$$|\alpha(l\mathbf{m})(\gamma_l)| \sim \frac{e^{l(\chi_{1,p}(\mathbf{m})-\chi_1(\mathbf{m}))}}{l}.$$

Thus h maps $\alpha(l\mathbf{m})(\gamma_l)$ into a segment of arbitrarily small length. But if $\chi_{1,p}(\mathbf{m}) - \chi_1(\mathbf{m}) > 0$ then the length of $|\alpha(l\mathbf{m})(\gamma_l)|$ tends to infinity and hence it converges to what we called $\tilde{\mathscr{P}}_1(p)$. As we saw, $\tilde{\mathscr{P}}_1(p)$ is mapped onto the fastest line at $h(p)$ which leads us to a contradiction. So $\chi_{1,p}(\mathbf{m}) - \chi_1(\mathbf{m}) \le 0$ and hence $\chi_{1,p}(\mathbf{m}) = \chi_1(\mathbf{m})$.

Finally, in order to prove that the other Lyapunov exponents coincide, just take an element of the action which has only one negative Lyapunov exponent, make this direction the fastest one and argue as with χ_1. $\square$

4. NON-ANOSOV ACTIONS WITH CARTAN HOMOTOPY DATA

Any action α of the type considered in this paper which contains an Anosov element is smoothly conjugate to the linear action α_0 [9].

The only known construction which produces non-Anosov examples of actions whose elements are homotopic to those of α_0 is based on "blowing up" a certain number of periodic orbits. This process is described in [5]. Let us consider the case when those orbits are fixed points. The outcome of this construction is an action on the torus with several spherical "holes" such that the action on each boundary component S is the projectivization β_0 of the linear action α_0 on $\mathbb{R}^{k+1}$. Furthermore, locally near each component the action looks as an $\mathbb{R}$-extension of the projective action with linear maps in the fibers. Outside of the boundary components the action is diffeomorphic to α_0 on $\mathbb{T}^{k+1}$ on the complement to the "blown up" fixed points. The size of the boundary component is one of the parameters of the construction and as it goes to zero the action converges to α_0 in C^0 topology but not in C^1.

In order to produce a $\mathbb{Z}^k$ action on the torus one just needs to describe how to fill the holes obtained as the result of the blowups. First, the action is extended as the skew product along the radii to the domain outside of a smaller concentric sphere S' in such a way that the radial motions are slowed down until they become C^∞ tangent to the identity. After that in the interior of the ball bounded by S' the action is defined on each concentric sphere via a smooth deformation of β_0 to the identity in the space of projective actions. This is of course possible because β_0 can be embedded into an $\mathbb{R}^k$ action generated by k commuting vector fields. The resulting action α_1 is isotopic to α_0 through smooth actions since the construction described above can be performed so that it depends continuously on the radius on the boundary component.

However, using arguments from [9] one can show that α_1 cannot be deformed to α_0 continuously in C^2 topology.

The following simple corollary of [1, Theorem 2.1] justifies consideration of such examples.

PROPOSITION 4.1. *If the semiconjugacy h is injective on an open set $\mathscr{O}$ then there exists a set S, the finite union of periodic orbits for α_0, such that h is a homeomorphism on the complement of $h^{-1}(S)$.*

Proof. Let $I \subset \mathbb{T}^{k+1}$ be the set of all points x for which $h^{-1}(h(x)) = x$. Since the preimage of any point is connected by Corollary 2.5, $I \supset \mathscr{O}$. Since h is a local homeomorphism near any point of $\mathscr{O}$, the set $h(I)$ which is α_0 invariant, contains an open and, by topological transitivity of α_0, an open dense set. Hence $S := h(\mathbb{T}^{k+1} \setminus I)$ is contained in a closed α_0-invariant set which by [1, Theorem 2.1] is finite. But since S is α_0-invariant it must be the union of finitely many periodic orbits.

Thus h is a homeomorphism between I and its image, which is the complement to a finite union of periodic orbits. □

REMARK 4.2. In the examples described above the semiconjugacy is actually a diffeomorphism outside of the preimages of a finite set of periodic orbits. It is not certain whether this is always true under the assumptions of Proposition 4, although our results imply the existence of derivatives in many directions at the preimage of a dense set.

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