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Local Rigidity of Partially Hyperbolic Actions. II: The Geometric Method and Restrictions of Weyl Chamber Flows on $SL(n, \mathbb{R})/\Gamma$

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We prove local differentiable rigidity for generic rank 2 restrictions of Weyl chamber flows by introducing a new "geometric" method in rigidity of actions of higher rank abelian groups based on the study of the combinatorial structure of the web of Lyapunov foliations.

1 Introduction

1.1 Actions of higher rank abelian groups and rigidity

In this paper, we make an essential step in realization of the program aiming at showing modified local differentiable rigidity for a broad class of algebraic (homogeneous and affine) partially hyperbolic actions of higher rank abelian groups. For a more detailed discussion of the rigidity program, see the introduction to [5]. For definitions and general background on partially hyperbolic dynamical systems [18]; all necessary background on algebraic actions can be found in [10]. We will also strongly rely on definitions, constructions, and results from our earlier paper [3].

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We consider algebraic actions of $\mathbb{Z}^k \times \mathbb{R}^l$, $k+l \geq 2$. The most general condition that leads to various rigidity phenomena (cocycle rigidity, local differentiable rigidity, measure rigidity, etc.) is the following:

> ($\mathfrak{R}$) *The group $\mathbb{Z}^k \times \mathbb{R}^l$ contains a subgroup L isomorphic to $\mathbb{Z}^2$ such that for the suspension of the restriction of the action to L every element other than identity acts ergodically with respect to the standard invariant measure obtained from the Haar measure.*

In the present paper, we treat a representative case of partially hyperbolic algebraic actions, satisfying condition ($\mathfrak{R}$), namely *restrictions of the Weyl chamber flow (WCF) on $SL(n,\mathbb{R})/\Gamma$*. Results of this paper has been announced in [4]. After this paper was written, the broad applicability and fruitfulness of this method have been amply demonstrated, see [2, 20, 21].

1.2 WCF on $SL(n,\mathbb{R})/\Gamma$ and generic restrictions

Let $X := SL(n,\mathbb{R})/\Gamma$ with $n \geq 4$ and Γ a cocompact lattice in $SL(n,\mathbb{R})$. Let $\mathbb{D}_+ = \{(t_1, \ldots, t_n) \in \mathbb{R}^n, \sum_{k=1}^n t_k = 0\}$ and let $D_+ = \exp \mathbb{D}_+$ be the group of diagonal matrices in $SL(n,\mathbb{R})$ with positive entries. The action by left translations of D_+ on X is the WCF, and we denote this action by α_0. For $0 \leq i \neq j \leq n$, $\mathbb{H}_{ij} = \{(t_1, \ldots, t_n) \in \mathbb{D}_+ : t_i = t_j\}$ are *Lyapunov hyperplanes* for the action α_0. Elements of $\mathbb{D}_+ \setminus \bigcup_{i \neq j} \mathbb{H}_{ij}$ are called *regular*. The connected components of the set of regular elements are *Weyl chambers*. The smallest nontrivial intersections of stable foliations of various elements of the action α_0 are *Lyapunov foliations*. Leaves of each Lyapunov foliation are one-dimensional. Each regular element either exponentially expands or exponentially contracts each of those leaves.

A two-dimensional plane $\mathbb{P} \subset \mathbb{D}_+$ is in *general position* if it intersects any two distinct Lyapunov hyperplanes along distinct lines. (It is for the reason of having *proper* planes of this kind that we assume $n \geq 4$ rather than $n \geq 3$ in the beginning of the current section.)

Let $\mathbb{G} \subset \mathbb{D}_+$ be a closed subgroup that contains a lattice $\mathbb{L}$ in a plane in general position and let $G = \exp \mathbb{G}$. One can naturally think of G as the image of an injective homomorphism $i_0 : \mathbb{Z}^k \times \mathbb{R}^l \to D_+$ (where $k+l \geq 2$). Restriction of the WCF to a subgroup G will be denoted by $\alpha_{0,G}$. The action $\alpha_{0,G}$ is referred to as a *generic restriction* of the WCF. It is given by

$$\alpha_{0,G}(a, x) = i_0(a) \cdot x. \tag{1.1}$$

1.3 Cocycles and rigidity

For an abelian group Y, a Y-valued cocycle over an action $\alpha : A \times M \to M$ is a continuous function $\beta : A \times M \to Y$ satisfying for any $a, b \in A$:

$$\beta(ab, x) = \beta(a, \alpha(b, x))\beta(b, x). \tag{1.2}$$

β is *cohomologous to a constant cocycle* if there exists a homomorphism $s : A \to Y$ and a continuous *transfer map* $H : M \to Y$ such that for all $a \in A$

$$\beta(a, x) = H(\alpha(a, x))s(a)H(x)^{-1}. \tag{1.3}$$

A cocycle is a *coboundary* if it is cohomologous to the *trivial cocycle* $\pi(a) = id_Y$, $a \in A$. For more detailed information on cocycles adapted to the present setting [3].

We proved in [3] that every Hölder (C^∞) cocycle with values in $\mathbb{R}^m$ over a generic restriction $\alpha_{0,G}$ is cohomologous to a constant cocycle via a continuous (C^∞) transfer map. Unlike proofs of previous cocycle rigidity results for algebraic actions of abelian groups, the proofs in [3] do not use harmonic analysis at all. Rather, we use the geometric structure of Lyapunov foliations of the action. In the present paper, we prove robustness of this approach and we use it to obtain local differentiable rigidity for $\alpha_{0,G}$. The method of [3] was further developed in [11] to cocycles with range in nonabelian and infinite-dimensional groups.

1.4 Formulation of results

Let $\alpha_{0,G}$ be a higher rank generic restriction of WCF with the acting group $\mathbb{Z}^k \times \mathbb{R}^l$, $k + l \geq 2$. Our main result is the following theorem.

Theorem 1.1 (Differentiable rigidity of generic restrictions). If $\tilde{\alpha}_G$ is a C^∞ action of $\mathbb{Z}^k \times \mathbb{R}^l$ sufficiently C^2-close to $\alpha_{0,G}$, then there exists a homomorphism $i : \mathbb{Z}^k \times \mathbb{R}^l \to D_+$ close to i_0 and a C^∞ diffeomorphism $h : X \to X$ such that $\tilde{\alpha}(a, h(x)) = h(i(a) \cdot x)$ for all $a \in \mathbb{Z}^k \times \mathbb{R}^l$. □

The principal ingredient in the proof of Theorem 1.1 is Theorem 1.2 which is the main technical result of the present paper. It generalizes the cocycle rigidity result from [3] to C^2-small perturbations of generic restrictions.

Theorem 1.2. Let $\tilde{\alpha}_G$ be a sufficiently C^2-small C^∞ perturbation of $\alpha_{0,G}$. If β is a Hölder cocycle over $\tilde{\alpha}_G$ with values in $\mathbb{R}^k$, then β is cohomologous to a constant cocycle given by a homomorphism $s : \mathbb{Z}^k \times \mathbb{R}^l \to D_+$ via a continuous transfer function. Furthermore, if the cocycle β is sufficiently small in a Hölder norm, the transfer map is C^0 arbitrary small. □

Other essential ingredients in the proof of Theorem 1.1 are Hirsch–Pugh–Shub stability theory [6], Theorem 6.1 from Section 6 which describes holonomy of the neutral foliation of the perturbed action along the Lyapunov foliations, and the old "a priori regularity" method for smoothness of the conjugacy. We prove Theorem 1.1 using these ingredients in the next section. The rest of the paper is dedicated to the proof of Theorem 1.2 and Theorem 6.1.

2 Cocycle Rigidity for Perturbations Implies Local Rigidity

2.1 Hölder conjugacy to perturbations along the leaves of the neutral foliation of $\alpha_{0,G}$

The neutral foliation $\mathcal{N}_0$ for $\alpha_{0,G}$ coincides with the orbit foliation of the WCF and is described in the introduction. Since it is a smooth foliation, we may use the Hirsch–Pugh–Shub structural stability theorem [6, Chapter 6]. Namely if $\tilde{\alpha}_G$ is a sufficiently C^1-small perturbation of $\alpha_{0,G}$, then for all $a \in A$ which are regular for $\alpha_{0,G}$ and sufficiently away from nonregular ones (denote this set by $\bar{A}$), the diffeomorphism $\tilde{\alpha}(a, \cdot)$ is also partially hyperbolic. The central distribution is the same for any $a \in \bar{A}$ and is uniquely integrable to an $\tilde{\alpha}(a, \cdot)$-invariant foliation which we denote by $\mathcal{N}$. Moreover, there is a Hölder homeomorphism $\tilde{h}$ of X, C^0 close to the id_X, which maps leaves of $\mathcal{N}_0$ to leaves of $\mathcal{N}$: $\tilde{h}\mathcal{N}_0 = \mathcal{N}$. This homeomorphism is uniquely defined in the transverse direction, that is, up to a homeomorphism preserving $\mathcal{N}$. Furthermore, $\tilde{h}$ can be chosen smooth and C^1 close to the identity along the leaves of $\mathcal{N}_0$.

Let us define an action α_G of G on X as the conjugate of $\tilde{\alpha}_G$ by the map $\tilde{h}$ obtained from the Hirsch–Pugh–Shub stability theorem:

$$\alpha_G := \tilde{h}^{-1} \circ \tilde{\alpha}_G \circ \tilde{h}.$$

Clearly, the leaves of the foliation $\mathcal{N}_0$ are preserved by every $a \in \bar{A}$. The action α_G is Hölder, but it is smooth and C^1-close to $\alpha_{0,G}$ along the leaves of the neutral foliation $\mathcal{N}_0$.

2.2 Proof of Theorem 1.1

Let $\tilde{\alpha}_G$ be a C^∞ action of $\mathbb{Z}^k \times \mathbb{R}^l$ close to $\alpha_{0,G}$ in C^2 topology and let α_G be the conjugate of $\tilde{\alpha}_G$ obtained via the Hirsch–Pugh–Shub homeomorphism $\tilde{h}$ as explained in Section 2.1.

Since the action α_G is a C^0 small perturbation of $\alpha_{0,G}$ along the leaves $\{D_+ \cdot x : x \in X\}$ of $\mathcal{N}_0$, we have that α_G is given by a map $\beta : (\mathbb{Z}^k \times \mathbb{R}^l) \times X \to D_+$ by

$$\alpha_G(a, x) = \beta(a, x) \cdot \alpha_{0,G}(a, x)$$

for $a \in \mathbb{Z}^k \times \mathbb{R}^l$ and $x \in X$. We will use multiplicative notation for the abelian group D_+ although it is isomorphic to $\mathbb{R}^{n-1}$ and we will apply Theorem 1.2 to cocycles with values in D_+. Notice that since α_G is a small perturbation of $\alpha_{0,G}$, it can be lifted to a G-action on $SL(n, \mathbb{R})$ commuting with the right Γ action on $SL(n, \mathbb{R})$, and β is a cocycle over α_G (for more details, see [14, Example 2.3]). In particular we have:

$$\beta(ab, x) = \beta(a, \alpha_G(b, x))\beta(b, x). \tag{2.1}$$

From this and Section 2.1 it follows that since α_G is Hölder, $\beta(a, x)$ is a small Hölder cocycle over the action α_G, due to the smallness of the perturbation. Thus, by Theorem 1.2, β is cohomologous to a small constant cocycle $s : \mathbb{Z}^k \times \mathbb{R}^l \to D_+$ via a continuous transfer map $H : X \to D_+$ which can be chosen close to the identity in C^0 topology if the perturbation $\tilde{\alpha}_G$ is small in C^2 topology.

Let us consider the map $h'(x) := H^{-1}(x) \cdot x$. We have from (2.1) and (1.3)

$$h'(\alpha_G(a, x)) = \alpha_{0,\tilde{G}}(a, h'(x))$$

where $\alpha_{0,\tilde{G}}(a, x) := i(a) \cdot x$, where $i(a) := s(a)i_0(a)$, $a \in A$ and i_0 is as in (1.1). Since the map h' is C^0 close to the identity, it is surjective and thus the action α_G is semi-conjugate to the standard perturbation $\alpha_{0,\tilde{G}}$ of $\alpha_{0,G}$, that is, $\alpha_{0,\tilde{G}}$ is a factor of α_G.

Proposition 2.1. The map h' is a homeomorphism that provides a topological conjugacy between α_G and $\alpha_{0,\tilde{G}}$. □

Proof. It is enough to prove that h' is injective. Since the map h' preserves the leaves of the foliation $\mathcal{N}_0$, the pre-image of any point belongs to a single leaf of that foliation. Furthermore, since h' is close to identity, the diameter of each such pre-image is small.

Now we pass to the (almost universal) cover $SL(n,\mathbb{R})$. Since the map h' is close to the identity on X, it is uniquely lifted to a close to the identity map on the cover for which we will use the same notation h'. Furthermore, if we show that the lifted map is injective on the cover, it will follow that the original $h'\colon X \to X$ is injective and hence a homeomorphism.

Next we show that if $h'(x) = h'(y)$ and $\mathcal{H}$ is an $\mathcal{F}$ holonomy, i.e., a product of holonomy maps between leaves of $\mathcal{N}_0$ within leaves of various center-Lyapunov foliations $\mathcal{W}_{ij}$, then $h'(\mathcal{H}(x)) = h'(\mathcal{H}(y))$. For the definition and discussion of foliations and holonomies, see Section 6. Obviously, it is sufficient to prove this for a holonomy $\mathcal{H}$ within a single leaf of $\mathcal{W}_{ij}$. But this follows immediately from the fact that the semi-conjugacy maps contracting manifolds of elements of α_G into contraction manifolds of corresponding elements of $\alpha_{0,\tilde{G}}$.

Since the $\mathcal{F}$ holonomy group acts transitively on the leaves of $\mathcal{N}_0$ if $h'(x) = h'(y)$, there is an $\mathcal{F}$ holonomy map F of the leaf $\mathcal{N}_0^x$ of $\mathcal{N}_0$ such that $F(x) = y$. Hence, $h'(F^n(x)) = h'(x)$ for any integer n, that is, h' maps the whole F orbit of x to the same point. But by Corollary 6.1 such orbits cannot have compact closure in the topology of the leaf. This contradiction proves that h' is injective. ■

Now by letting $h := h'\tilde{h}^{-1}$ we have that $h \circ \tilde{\alpha}_G \circ h^{-1} = \alpha_{0,\tilde{G}}$. Thus there is a *topological* conjugacy between $\tilde{\alpha}_G$ and a standard perturbation of $\alpha_{0,G}$. In particular, the conjugacy takes Lyapunov foliations of $\alpha_{0,G}$ into those of $\tilde{\alpha}_G$. Proving further that the conjugacy is smooth along the leaves of Lyapunov foliations of $\alpha_{0,G}$ follows by an application of the Katok–Spatzier method of nonstationary normal forms [12, Corollary 10 and Section 2.2.2]. The smoothness of $\tilde{h}$ then follows just as in [12] from the fact that Lyapunov directions for $\alpha_{0,G}$ with their Lie brackets span the tangent space at every point.

3 Lyapunov Cycles and Cocycle Rigidity

3.1 Preliminaries

Let $\alpha : A \to \mathrm{Diff}(M)$ be an action of $A := \mathbb{Z}^k \times \mathbb{R}^l$ on a compact manifold M by diffeomorphisms of M preserving an ergodic probability measure μ. Then there are finitely many linear functionals λ on A, called *Lyapunov exponents*, a set of full measure Λ and a measurable splitting of the tangent bundle $T_\Lambda M = \bigoplus_\lambda E^\lambda$, such that for $v \in E^\lambda$ and $a \in A$ the Lyapunov exponent of v with respect to $\alpha(a)$ is $\lambda(a)$.

If χ is a nonzero Lyapunov exponent, then we define its *coarse Lyapunov subspace* by

$$E_\chi := \bigoplus_{\{\lambda = c\chi : c>0\}} E^\lambda.$$

For every $a \in A$, one can define stable, unstable, and neutral subspaces for a by $E_a^s = \bigoplus_{\lambda(a)<0} E^\lambda$, $E_a^u = \bigoplus_{\lambda(a)>0} E^\lambda$, and $E_a^0 = \bigoplus_{\lambda(a)=0} E^\lambda$. In particular, for any $a \in A := \bigcap_{\chi \neq 0} (\operatorname{Ker} \chi)^c$, the subspace E_a^0 is the same and thus can be denoted simply by E^0. Hence, for any such a, there is a splitting: $TM = E_a^s \oplus E^0 \oplus E_a^u$.

If, in addition, E^0 is a continuous distribution uniquely integrable to a foliation $\mathcal{N}$ with smooth leaves, and if there exists $a \in A$ such that $\alpha(a)$ is uniformly normally hyperbolic with respect to $\mathcal{N}$ (in the sense of the Hirsch–Pugh–Shub [6]) then α is a *partially hyperbolic action*. The elements in A which are uniformly normally hyperbolic with respect to $\mathcal{N}$ are called *regular*. Let $\tilde{A}$ be the set of regular elements.

If the set $\tilde{A}$ is dense in $\mathbb{R}^k$, then for each nonzero Lyapunov exponent χ and every $p \in M$ the coarse Lyapunov distribution is:

$$E_\chi(p) = \bigcap_{\{a \in \tilde{A} : \chi(a)<0\}} E_a^s(p).$$

The right-hand side is Hölder and can be extended to a Hölder distribution tangent to the foliation $\mathcal{T}_\chi := \bigcap_{\{a \in \tilde{A} : \chi(a)<0\}} \mathcal{W}_a^s$ with C^∞ leaves. This is the *coarse Lyapunov foliation* corresponding to χ [3, Section 2].

We denote by $\chi_1, \dots, \chi_r$ a maximal collection of nonzero Lyapunov exponents that are not positive multiples of one another and by $\mathcal{T}_1, \dots, \mathcal{T}_r$ the corresponding coarse Lyapunov foliations. Given a foliation $\mathcal{T}_i$ and $x \in M$, we denote by $\mathcal{T}_i(x)$ the leaf of $\mathcal{T}_i$ through x.

3.2 Paths and cycles for a collection of foliations

Let $\mathcal{T}_1, \dots, \mathcal{T}_r$ be a collection of mutually transversal continuous foliations on M, with smooth simply connected leaves.

Definition 3.1. For $N \in \mathbb{N}$ and $j_k \in \{1, \dots, r\}$, $k \in \{1, \dots, N-1\}$, an ordered set of points $\mathfrak{p}(j_1, \dots, j_{N-1}) : x_1, \dots, x_N \in M$ is called an *$\mathcal{T}$-path* of length N if for every $k \in \{1, \dots, N-1\}$, $x_{i+1} \in \mathcal{T}_{j_k}(x_k)$. □

Definition 3.2. For $N \in \mathbb{N}$ and $j_k \in \{1, \ldots, r\}$, $k \in \{1, \ldots, N\}$, an ordered set of points $\mathfrak{c}(j_1, \ldots, j_N) : x_1, \ldots, x_N, x_{N+1} = x_1 \in M$ is called a *$\mathcal{T}$-cycle* of length N if for every $k \in \{1, \ldots, N\}$, $x_{k+1} \in \mathcal{T}_{j_k}(x_k)$. A $\mathcal{T}$-cycle which consists of a single point is a *trivial $\mathcal{T}$-cycle*. □

Remark. We will denote a $\mathcal{T}$-cycle $\mathfrak{c}(j_1, \ldots, j_N)$ by $\mathfrak{c}$ whenever the short notation causes no confusion. □

Let $\mathcal{C}(\mathcal{T})$ denote the collection of $\mathcal{T}$-cycles. For $x \in X$, let $\mathcal{C}^x(\mathcal{T})$ the collection of $\mathcal{T}$-cycles with an initial point x. In $\mathcal{C}^x(\mathcal{T})$, cycles $\mathfrak{c}(j_1, \ldots, j_N) : x_1, \ldots, x_k, x_k, \ldots, x_N, x_{N+1} = x_1$ and $\mathfrak{c}(j_1, \ldots, j_N) : x_1, \ldots, x_k, \ldots, x_N, x_{N+1} = x_1$ are identified. In particular, a cycle $x, x, x, \ldots, x$ is identified with the cycle $\mathfrak{o} : x$, which is called a *trivial cycle*.

Now we introduce natural operations in $\mathcal{C}(\mathcal{T})$ modeled on the operations on loops which appear in the definition of the fundamental group.

- For two cycles in $\mathcal{C}^x(\mathcal{T})$, $\mathfrak{c}(j_1, \ldots, j_N) : x = x_1, \ldots, x_N, x_{N+1} = x$ and $\mathfrak{c}'(j'_1, \ldots, j'_N) : x = x'_1, \ldots, x'_N, x'_{N+1} = x$ define their *composition* $\mathfrak{c} * \mathfrak{c}'$ by

$$\mathfrak{c} * \mathfrak{c}'(j_1, \ldots, j_N, j'_1, \ldots, j'_N) : x = x_1, \ldots, x_N, x'_1, \ldots, x'_N, x'_{N+1} = x.$$

- The *inverse* of a cycle $\mathfrak{c} := \mathfrak{c}(j_1, \ldots, j_N) : x_1, \ldots, x_N, x_{N+1} = x_1$ is the cycle $\mathfrak{c}^{-1} := \mathfrak{c}^{-1}(j_N, \ldots, j_1) : x_1, x_N, \ldots, x_2, x_1$.
- Let $\mathfrak{c}(j_1, \ldots, j_N) : x_1, \ldots, x_N, x_{N+1} = x_1 \in M$ be a $\mathcal{T}$-cycle and let $y_1 \in \mathcal{T}_j(x_1)$ for some $j \in \{1, \ldots, r\}$. Then we call the cycle

$$\mathfrak{c}'(j, j_1, \ldots, j_N, j) : y_1, x_1, x_2, \ldots, x_N, x_{N+1} = x_1, y_1$$

  a *conjugate* (or a $\mathcal{T}_j$-conjugate) of $\mathfrak{c}(j_1, \ldots, j_N)$.
- Let $\mathfrak{c}(j_1, \ldots, j_N) : x_1, x_2, \ldots, x_N, x_{N+1} = x_1 \in \mathcal{C}(\mathcal{T})$, and $\mathfrak{c}_n(j_1, \ldots, j_N) : x_1^{(n)}, x_2^{(n)}, \ldots, x_N^{(n)}, x_{N+1}^{(n)} = x_1^{(n)}$. Then $\mathfrak{c} = \lim_{n\to\infty} \mathfrak{c}_n$ if for all $k \in \{1, \ldots, N\}$, $x_k = \lim_{n\to\infty} x_k^{(n)}$.

Definition 3.3. A $\mathcal{T}$-cycle $\mathfrak{c}(j_1, \ldots, j_N) : x_1, \ldots, x_N, x_{N+1} = x_1 \in M$ is *contractible*, if there is a closed path $\tilde{\mathfrak{c}}$ in M obtained by connecting for each $k = 1, \ldots, N$ the points x_k and x_{k+1} by a path on the leaf of the foliation $\mathcal{T}_{j_k}$, and $\tilde{\mathfrak{c}}$ is contractible. □

Notice that the class of contractible cycles is closed under operations described above: composition, taking inverse, conjugation, and taking limit.

3.3 Stable cycles, allowable substitutions, and reducible cycles

Let $\mathcal{T}_1, \ldots, \mathcal{T}_r$ be the coarse Lyapunov foliations with smooth leaves of a partially hyperbolic action α on M. Notice that under this assumption the leaves of $\mathcal{T}_i$ for each i are simply connected, because every loop within a leaf is mapped by a diffeomorphism into a loop of an arbitrary small diameter, and hence contractible, inside another leaf.

Definition 3.4. A $\mathcal{T}$-cycle $\mathfrak{c}$ is called *stable* for the A action α if there exists a regular element $a \in \mathbb{R}^k$ such that $x_k \in \mathcal{W}^s_a(x_1)$ for all $k \in \{1, \ldots, N\}$. Denote by $\mathcal{AS}^s_{\mathcal{T}}(\alpha)$ the collection of such cycles. □

Notice that every stable cycle is contractible since it is mapped by a diffeomorphism to a cycle of arbitrarily small diameter. Simplest examples of stable cycles are cycles contained in a leaf of some $\mathcal{T}_j$ foliation.

Definition 3.5. A path $\mathfrak{p}$ *reduces* to a path $\mathfrak{p}'$ with the same endpoints as $\mathfrak{p}$, via an *α-allowable $\mathcal{T}$-substitution of s-type* if the $\mathcal{T}$-cycle $\mathfrak{p} * \mathfrak{p}'$ obtained by concatenation of $\mathfrak{p}$ and $\mathfrak{p}'$ is a stable $\mathcal{T}$-cycle. □

We denote by $\mathcal{AS}^{rs}_{\mathcal{T}}(\alpha)$ the collection of all $\mathcal{T}$-cycles that reduce to a trivial cycle, that is, to a point, via finitely many α-allowable $\mathcal{T}$-substitutions of s-type. In particular, $\mathcal{AS}^{rs}_{\mathcal{T}}(\alpha)$ contains conjugates of all cycles in $\mathcal{AS}^s_{\mathcal{T}}(\alpha)$. Elements of $\mathcal{AS}^{rs}_{\mathcal{T}}(\alpha)$ we also refer to as substitutions (or allowable substitutions) of rs-type.

Let $\mathcal{AS}_{\mathcal{T}}(\alpha)$ denote the collection of $\mathcal{T}$-cycles which contains $\mathcal{AS}^{rs}_{\mathcal{T}}(\alpha)$ and is closed in $\mathcal{C}(\mathcal{T})$ (under the limiting procedure, Section 3.2).

Definition 3.6. A path $\mathfrak{p}$ *reduces* to a path $\mathfrak{p}'$ with the same endpoints as $\mathfrak{p}$, via an *α-allowable $\mathcal{T}$-substitution* if the $\mathcal{T}$-cycle $\mathfrak{p} * \mathfrak{p}'$ obtained by the concatenation of $\mathfrak{p}$ and $\mathfrak{p}'$ is in the collection $\mathcal{AS}_{\mathcal{T}}(\alpha)$. Accordingly, cycles in $\mathcal{AS}_{\mathcal{T}}(\alpha)$ are called *α-allowable $\mathcal{T}$-substitutions*. □

Remarks.

(1) We will sometimes use the notation $\mathcal{AS}_{\mathcal{T}}(\alpha)^x$ (corr. $\mathcal{AS}^s_{\mathcal{T}}(\alpha)^x$, $\mathcal{AS}^{rs}_{\mathcal{T}}(\alpha)^x$) for $\mathcal{T}$ cycles in $\mathcal{AS}_{\mathcal{T}}(\alpha)$ (corr. $\mathcal{AS}^s_{\mathcal{T}}(\alpha)$, $\mathcal{AS}^{rs}_{\mathcal{T}}(\alpha)$) with initial point x.

(2) Since the leaves of $\mathcal{T}_i$ for all $i \in \{1, \ldots, r\}$ are simply connected, the cycles in $\mathcal{AS}_{\mathcal{T}}(\alpha)$ are contractible.

(3) Paths and cycles for a collection $\mathcal{T}$ of foliations which are Lyapunov foliations for a partially hyperbolic action we often refer to as to Lyapunov paths and Lyapunov cycles. □

Definition 3.7. Two $\mathcal{T}$-cycles $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are *α-equivalent*, if $\mathfrak{c}_1$ reduces to $\mathfrak{c}_2$ via a finite sequence of α-allowable $\mathcal{T}$-substitutions. A $\mathcal{T}$-cycle is called *α-reducible*, if it is α-equivalent to a trivial $\mathcal{T}$-cycle, that is, if it can be reduced to a point via finitely many α-allowable $\mathcal{T}$-substitutions. Clearly, all $\mathcal{T}$-cycles in $\mathcal{AS}_{\mathcal{T}}(\alpha)$ are $\mathcal{T}$-reducible. □

The set of $\mathcal{T}$-cycles $\mathcal{C}^x(\mathcal{T})$ with an initial point $x \in X$ factored by the relation of α-equivalence is denoted by $\mathcal{R}(\alpha, \mathcal{T})^x$. It clearly has a group structure under the operation induced by concatenation of $\mathcal{T}$-cycles at x.

The following lemma immediately follows from the fact that every $\mathcal{T}$-cycle of the type $\mathfrak{c}: x, y, x$ where $y \in \mathcal{T}_j(x)$ for some $j \in \{1, \ldots, r\}$, is a stable $\mathcal{T}$-cycle and thus is α-reducible.

Lemma 3.1. For every $i \in \{1, \ldots, r\}$ and $x \in M$, $y \in \mathcal{T}_i(x)$, the groups $\mathcal{R}(\alpha, \mathcal{T})^x$ and $\mathcal{R}(\alpha, \mathcal{T})^y$ are isomorphic. □

3.4 Transitivity of foliations

For a submanifold Y in M, $d_Y(x, y)$ denotes the infimum of lengths of smooth curves in Y connecting x and y.

Definition 3.8. A collection $\mathcal{T}$ of foliations $\mathcal{T} = \{\mathcal{T}_1, \ldots, \mathcal{T}_r\}$ is called *transitive*, if there exist $N \in \mathbb{N}$ and $R > 0$ such that any two points $x, y \in X$ can be connected by a $\mathcal{T}$-path $\mathfrak{p}(j_1, \ldots, j_{N-1}) : x_1 = x, x_2, \ldots, x_N = y$ such that $x_{k+1} \in \mathcal{T}_{j_k}(x_k)$ and $d_{\mathcal{T}_{j_k}(x_k)}(x_{k+1}, x_k) < R$. □

Lemma 3.2. If $\mathcal{T}$ is a transitive collection of coarse Lyapunov foliations of a partially hyperbolic action α, then, for $x \in M$, $\mathcal{R}(\alpha, \mathcal{T})^x$ are all isomorphic and hence can be denoted by $\mathcal{R}(\alpha, \mathcal{T})$. □

Proof. For x and y in M due to transitivity of the collection $\mathcal{T}$ of foliations $\mathcal{T}_1, \ldots, \mathcal{T}_r$, there exists a $\mathcal{T}$-path $\mathfrak{p}(j_1, \ldots, j_{N-1}) : x = x_1, x_2, \ldots, x_{N-1}, x_N = y$. Now, because of the invariance in Lemma 3.1, we have that $\mathcal{R}(\alpha, \mathcal{T})^x_1 = \mathcal{R}(\alpha, \mathcal{T})^{x_2} = \cdots = \mathcal{R}(\alpha, \mathcal{T})^{x_N}$. Thus $\mathcal{R}(\alpha, \mathcal{T})^x = \mathcal{R}(\alpha, \mathcal{T})^y$. ■

We remark that as in the case of the fundamental group, the isomorphism between $\mathcal{R}(\alpha, \mathcal{T})^x$ and $\mathcal{R}(\alpha, \mathcal{T})^y$ need not be canonical.

Definition 3.9. Foliations $\mathcal{T}_1, \ldots, \mathcal{T}_r$ are *locally transitive* (denoted this property by LT), if there exists $N \in \mathbb{N}$ such that for any $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in M$ and for every $y \in B_M(x, \delta)$ (where $B_M(x, \delta)$ is δ ball in M) there is a $\mathcal{T}$-path $\mathfrak{p}(j_1, \ldots, j_{N-1}) : x = x_1, x_2, \ldots, x_{N-1}, x_N = y$ in the ball $B_M(x, \epsilon)$ such that $x_{k+1} \in \mathcal{T}_{j_k}(x_k)$ and $d_{\mathcal{T}_{j_k}(x_k)}(x_{k+1}, x_k) < 2\epsilon$. □

Remark. Both transitivity properties defined above are preserved under a Hölder conjugacy. The same holds true for the group $\mathcal{R}(\alpha, \mathcal{T})$ when it is well defined, that is, when $\mathcal{T}$ is a transitive collection of Lyapunov foliations for an action α. Notice that local transitivity implies transitivity [1]. □

3.5 Reducibility and cocycle rigidity

Two propositions in this section are easy consequences of [3, Proposition 4] and our definition of α-reducible cycles.

Definition 3.10. For a partially hyperbolic A-action α on a compact manifold M with coarse Lyapunov foliations $\mathcal{T}_1, \ldots, \mathcal{T}_r$ and for a cocycle $\beta : A \times M \to Y$ over α, where Y is an abelian Lie group, we define Y-valued potential of β as

$$P_a^j(y, x) = \lim_{n \to +\infty} \beta(na, y)^{-1} \beta(na, x), \qquad \chi_j(a) < 0 \tag{3.1}$$

$$P_a^j(y, x) = \lim_{n \to -\infty} \beta(na, y)^{-1} \beta(na, x), \qquad \chi_j(a) > 0 \tag{3.2}$$

where a is a regular element in A, $j \in \{1, \ldots, r\}$, $x \in M$ and $y \in \mathcal{T}_j(x)$.

For any $\mathcal{T}$-cycle $\mathfrak{c} : x_1, \ldots, x_{N+1} = x_1$ on M, we define the corresponding periodic cycle functional (PCF):

$$PCF(\mathfrak{c})(\beta) = \prod_{i=1}^{N} P_a^{j(i)}(x_i, x_{i+1})(\beta). \tag{3.3}$$

It is proved in [3] that the expression for PCF does not depend on the choice of a. □

Two essential properties of the PCF which are crucial for our purpose are that PCF is continuous and that it is invariant under the operation of moving cycles around by elements of the action α. The latter property has an immediate consequence that PCF vanishes on all stable cycles and all cycles which can be reduced to a trivial cycle via α-allowable $\mathcal{T}$-substitutions of s-type or rs-type. The former allows us to consider limits of cycles and implies that PCF vanishes on any cycle which can be reduced to a trivial one via α-allowable substitutions, that is, on any α-reducible cycle.

Proposition 3.1. If $\mathcal{T}:\mathcal{T}_1,\dots,\mathcal{T}_r$ is an LT collection of coarse Lyapunov foliations for a partially hyperbolic A-action α and if the group $\mathcal{R}(\alpha,\mathcal{T})$ has no nontrivial homomorphism into $\mathbb{R}^l$, then every Hölder $\mathbb{R}^l$-valued cocycle over α is cohomologous to a constant cocycle via a continuous transfer function.

If the cocycle is smooth, then the transfer map is smooth along the leaves of the coarse Lyapunov foliations in $\mathcal{T}$. □

Proof. Let $\mathcal{C}^x(\mathcal{T})$ be the collection of $\mathcal{T}$-cycles with initial point $x\in M$. If $f:\mathcal{C}^x(\mathcal{T})\to\mathbb{R}^l$ is a map such that: (1) $f(\mathfrak{c}*\mathfrak{c}_1)=f(\mathfrak{c})+f(\mathfrak{c}_1)$ for $\mathfrak{c},\mathfrak{c}_1\in\mathcal{C}^x(\mathcal{T})$, where $*$ denotes concatenation of cycles at point x, (2) f vanishes on all stable $\mathcal{T}$-cycles, and (3) f is continuous, then, by the definition of α-reducible $\mathcal{T}$-cycles we have that f vanishes on all α-reducible $\mathcal{T}$-cycles in $\mathcal{C}^x(\mathcal{T})$. Thus f defines a homomorphism from $\mathcal{R}(\alpha,\mathcal{T})$ into $\mathbb{R}^l$ which by assumption has to be trivial. Therefore, f vanishes on all $\mathcal{T}$-cycles in $\mathcal{C}^x(\mathcal{T})$ and since x is arbitrary, on all $\mathcal{T}$-cycles.

Now we use [3, Proposition 4]. The PCF which appears in [3, Proposition 4] satisfies conditions (1), (2), and (3) ([3, Section 3]), thus by the above analysis the PCF vanishes on all $\mathcal{T}$-cycles. This implies trivialization of $\mathbb{R}^l$-valued cocycles over α by [3, Proposition 4]. ■

Proposition 3.2. Let $\mathcal{T}:\mathcal{T}_1,\dots,\mathcal{T}_r$ be an LT collection of coarse Lyapunov foliations of a partially hyperbolic A-action α. Suppose that

(1) If $\mathcal{T}$-cycle $\mathfrak{c}$ is contractible, then there exists $n\in\mathbb{N}$ such that $\mathfrak{c}^n$ is α-reducible.
(2) $\pi_1(M)$ has no nontrivial homomorphism into $\mathbb{R}^l$,

then every $\mathbb{R}^l$-valued Hölder cocycle over α is cohomologous to a constant cocycle. If the cocycle is smooth, then the transfer map is smooth along the leaves of the foliations in $\mathcal{T}$. If the cocycle is small on a compact set of generators of A, uniformly on M in some Hölder norm, then the transfer map is C^0 close to the trivial one. □

Proof. By assumption a power of any contractible $\mathcal{T}$-cycle is α-reducible, and the PCF vanishes on all α-reducible cycles, thus the PCF vanishes on a power of any contractible $\mathcal{T}$-cycle. Since PCF takes the concatenation of two cycles to the sum of their corresponding PCFs, it follows that PCF vanishes on every contractible $\mathcal{T}$-cycle. Hence PCF induces a homomorphism from $\pi_1(X)$ into $\mathbb{R}^l$. Since it has to be trivial by assumption, PCF is trivial on every $\mathcal{T}$-cycle. Thus the claim follows from [3, Proposition 4].

In particular, if the cocycle is small on generators in some Hölder norm, then due to local transitivity of $\mathcal{T}$-foliations and from the computation in [3, Proposition 2], the transfer map obtained by this construction is C^0 close to the trivial one. ■

4 Lyapunov Cycles and Allowable Substitutions for $\alpha_{0,G}$

4.1 Lyapunov foliations for α_0 and $\alpha_{0,G}$

Let $e_{ij}(t)$, $1 \le i,\ j \le n,\ i \ne j$ be the standard unipotent one-parameter subgroups in $SL(n,\mathbb{R})$ and let $\mathcal{U}_{ij}$ be the corresponding homogeneous unipotent foliations of X. For $x \in X = SL(n,\mathbb{R})/\Gamma$, the leaf of $\mathcal{U}_{ij}$ through x is $\mathcal{U}_{ij}^x = \{e_{ij}(t)x | t \in \mathbb{R}\}$. These foliations are invariant under the action by left translations of any element in D_+ and are the coarse Lyapunov foliations for α_0, that is, for the full WCF. If $\mathbb{P}$ is a two-plane in general position, then the foliations $\mathcal{U}_{ij}$, $1 \le i,\ j \le n, i \ne j$ are also coarse Lyapunov foliations for $\alpha_{0,P}$. The leaves of $\mathcal{U}_{ij}$ are intersections of the leaves of stable manifolds of the action by different elements of $\mathbb{P}$. See [3, Section 5.2] for details. The same holds for the action by any regular lattice in $\mathbb{P}$ and thus for any generic restriction $\alpha_{0,G}$. We denote by $\mathcal{U}$ the collection $\mathcal{U}_{ij}$, $1 \le i,\ j \le n, i \ne j$. The neutral foliation for $\alpha_{0,G}$ is $\mathcal{N}_0$, the orbit foliation of the full WCF.

The following is a simple consequence of the fact that the foliations $\mathcal{U}_{ij}$ are totally nonintegrable of index 2 (i.e., the vector fields tangent to $\mathcal{U}_{ij}$, $1 \le i \ne j \le n$ and their Lie brackets already span the tangent space at any $x \in X$), see [1, Theorem 4.2] or [9, Proposition 1].

Proposition 4.1. Foliations $\mathcal{U}_{ij}$ are locally $\frac{1}{2}$-Hölder transitive. □

4.2 Elementary Lyapunov cycles for $\alpha_{0,G}$

One consequence of the Steinberg description of central extensions [19] is the following:

Steinberg Theorem. The group $SL(n,\mathbb{R})$ is generated by unipotent elements $e_{ij}(t)$, $t\in\mathbb{R}$, $1\le i\ne j\le n$ subject to relations:

$$e_{ij}(t)e_{ij}(s)=e_{ij}(t+s) \tag{4.1}$$

$$[e_{ij}(t),e_{kl}(s)]=\begin{cases}1, & j\ne k, i\ne l\\ e_{il}(st), & j=k, i\ne l\\ e_{kj}(-st), & k\ne j, i=l\end{cases} \tag{4.2}$$

$$h_{12}(t)h_{12}(s)h_{12}(ts)^{-1}=1 \tag{4.3}$$

where $h_{12}(t):=e_{12}(t)e_{21}(-t^{-1})e_{12}(t)e_{12}(-1)e_{21}(1)e_{12}(-1)$ for each $t\in\mathbb{R}^*$. □

We call the $\mathcal{U}$-cycles induced by the relations above, *elementary* $\mathcal{U}$-cycles, and those induced by (4.3) we call *diagonal* cycles. The structure of diagonal cycles is well understood: set of all diagonal cycles at a point x with the operation of concatenation of cycles at a point x is related to the group $K_2\mathbb{R}$ which has the following presentation [15].

Matsumoto Theorem. The group $K_2\mathbb{R}$ is generated by symbols $\{s,t\}$, $s,t\in\mathbb{R}^*$ subject to relations:

1. $\{s',t\}\{s,t\}=\{s's,t\},\ \{s,t\}\{s,t'\}=\{s,tt'\}$
2. $\{s,1-s\}=1,\ s\ne 1$
3. $\{s,-s\}=1$

where R^* denotes the multiplicative group of nonzero real numbers. □

The following lemma is crucial for the method developed in this paper. Namely, it demonstrates that the contractible diagonal cycles even when not reducible to stable are in fact limits of $\mathcal{U}$-cycles that are reducible to stable ones.

Lemma 4.1. Each elementary $\mathcal{U}$-cycle, if contractible, induces an $\alpha_{0,G}$-allowable $\mathcal{U}$-substitution:

(1) $\mathcal{U}$-cycles induced by relations (4.1) and (4.2) are in $\mathcal{AS}^s_{\mathcal{U}}(\alpha_{0,G})$.

(2) $\mathcal{U}$-cycles induced by relations (4.3) with $s\ge 0$ or $t\ge 0$ are in $\mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})$.

(3) $\mathcal{U}$-cycles induced by relations (4.3) with s, t both negative, can be reduced via $\alpha_{0,G}$-allowable $\mathcal{T}$-substitutions of rs-type to a $\mathcal{U}$-cycle induced by the relation $\{-1,-1\}$. If doubled, those cycles are also in $\mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})$. □

Proof. (1) From the assumption that $\alpha_{0,G}$ is a *generic* restriction of the WCF (see [3, Proposition 7]), we have that $\mathcal{AS}^s_{\mathcal{U}}(\alpha_{0,G}) = \mathcal{AS}^s_{\mathcal{U}}(\alpha_0)$. In particular, all commutator $\mathcal{U}$-cycles induced by relations (4.1) and (4.2) are stable for $\alpha_{0,G}$ and thus are contained in $\mathcal{AS}^s_{\mathcal{U}}(\alpha_{0,G})$.

(2) Each generator $\{s,t\}$ of the group $K_2\mathbb{R}$ induces at any given point $x \in X$ a $\mathcal{U}$-cycle. Call it $\mathfrak{c}^x_{\{s,t\}}$. It is a $\mathcal{U}$-cycle that consists of 18 points and is induced by the corresponding relation $h_{12}(s)h_{12}(t)h_{12}(st)^{-1} = 1$ in (4.3) with the initial point x, where $h_{12}(t)$ is as in (4.3).

Relations 1., 2., and 3. in Matsumoto Theorem are proved by using only relations (4.1), (4.2), and/or their conjugates, see for example [16, Chapter 11]. This implies that $\mathcal{U}$-cycles $\mathfrak{c}^x_{\{s,t\}\{s,t'\}\{s,tt'\}^{-1}}$, $\mathfrak{c}^x_{\{s,t\}\{s',t\}\{ss',t\}^{-1}}$, $\mathfrak{c}^x_{\{s,1-s\}}$ and $\mathfrak{c}^x_{\{s,-s\}}$ are in $\mathcal{AS}^{rs}_{\mathcal{U}}(\alpha_{0,G})$.

In particular, since $\mathfrak{c}^x_{\{s,t\}\{s,t'\}\{s,tt'\}^{-1}} = \mathfrak{c}^x_{\{s,t\}} * \mathfrak{c}^x_{\{s,t'\}} * \mathfrak{c}^x_{\{s,tt'\}^{-1}}$, this implies that, if $\mathfrak{c}^x_{\{s,t\}}$ and $\mathfrak{c}^x_{\{s,t'\}}$ are in $\mathcal{AS}^{rs}_{\mathcal{U}}(\alpha_{0,G})$, then so is $\mathfrak{c}^x_{\{s,tt'\}}$. Similarly, if $\mathfrak{c}^x_{\{s,t\}}$ and $\mathfrak{c}^x_{\{s',t\}}$ are in $\mathcal{AS}^{rs}_{\mathcal{U}}(\alpha_{0,G})$, then so is $\mathfrak{c}^x_{\{ss',t\}}$. This combined with the fact that cycles $\mathfrak{c}^x_{\{3,-3\}}$ and $\mathfrak{c}^x_{\{3,-2\}}$ are in $\mathcal{AS}^{rs}_{\mathcal{U}}(\alpha_{0,G})$ (due to 2. and 3. of Matsumoto Theorem) implies that all $\mathfrak{c}^x_{\{3,(-3)^i(-2)^j\}}$ are in $\mathcal{AS}^{rs}_{\mathcal{U}}(\alpha_{0,G})$. This holds true not only for positive powers i and j but also for negative ones by using inverses in the group $K_2\mathbb{R}$.

Since the sequence $\{(-3)^i(-2)^j\}_{i,j}$ is dense, we have that $\mathfrak{c}_{\{3,t\}}$ for every $t \in \mathbb{R}^*$ is a $\mathcal{U}$-cycle which is a limit of elements in $\mathcal{AS}^{rs}_{\mathcal{U}}(\alpha_{0,G})$, so it is in $\mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})$.

Similarly, starting with $\mathfrak{c}^x_{\{4,-3\}} \in \mathcal{AS}^{rs}_{\mathcal{U}}(\alpha_{0,G})$ and $\mathfrak{c}^x_{\{4,-4\}} \in \mathcal{AS}^{rs}_{\mathcal{U}}(\alpha_{0,G})$ (again due to 2. and 3. of Matsumoto Theorem), we get $\mathfrak{c}^x_{\{4,t\}}$ is in $\mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})$ for every $t \in \mathbb{R}^*$. Thus every $\mathfrak{c}^x_{\{3^i4^j,t\}}$ is in $\mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})$ and so is any limit, that is, any $\mathfrak{c}^x_{\{s,t\}}$, for $s > 0$. In a similar way, one obtains $\mathfrak{c}^x_{\{s,t\}} \in \mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})$, $t > 0$. This proves part (2). Proof of part (3) is similar and we omit it.

This proof is an analogue for cycles of Milnor's proof in [16, Theorem A1] that continuous Steinberg symbols over $\mathbb{R}^*$ have an order 2 . ■

4.3 Lyapunov cycles for $\alpha_{0,G}$

Any $\mathcal{U}$-path which consists of N points is determined by its initial point x and an ordered sequence of unipotent elements $e_{i_k j_k}(t_k)$, $k = 1, \ldots, N$.

Every contractible $\mathcal{U}$-cycle is represented by a relation

$$e_{i_1 j_1}(t_1), \ldots, e_{i_N j_N}(t_N) = 1 \tag{4.4}$$

in the group $SL(n,\mathbb{R})$. Call this $\mathcal{U}$-cycle $\mathfrak{u}$. Using the presentation of $SL(n,\mathbb{R})$ from the Steinberg Theorem, every relation of the type (4.4) above is a product of relations (4.1),(4.2), (4.3), and/or their conjugates. If $\mathfrak{u}$ is contractible, then the product involves elements with s and t not both negative from (4.3). Therefore, via elementary $\mathcal{U}$-cycles $\mathfrak{u}$ can be reduced to the trivial cycle that is, to a point. From Lemma 4.1, all elementary $\mathcal{U}$-cycles of this type and their conjugates are in $\mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})$. Since the leaves of $\mathcal{U}_{ij}$-foliations are simply connected, we have the following:

Proposition 4.2. For a generic restriction $\alpha_{0,G}$, all contractible $\mathcal{U}$-cycles are $\alpha_{0,G}$-reducible and vice versa, hence $\mathcal{R}(\alpha_{0,G},\mathcal{U})$ is isomorphic to $\pi_1(X)$. □

Now let $\tilde{\alpha}_G$ be a C^2-small smooth perturbation of a generic restriction $\alpha_{0,G}$. The goal of the next two sections is to establish that the structure of Lyapunov cycles for $\tilde{\alpha}_G$ is not very different from that of $\alpha_{0,G}$.

5 Preliminaries on Perturbations of $\alpha_{0,G}$

5.1 Lyapunov foliations for perturbations of generic restrictions

From this point on we assume α_G is a Hölder perturbation of $\alpha_{0,G}$ along the leaves of the neutral foliation $\mathcal{N}_0$, which is smooth along the leaves of that foliation. (See Section 2.1.)

Denote the collection of Lyapunov foliations for $\tilde{\alpha}_G$ by $\tilde{\mathcal{F}}$. Since $\tilde{\alpha}_G$ and α_G are conjugate via a Hölder map, the stable and unstable, and thus the corresponding Lyapunov foliations are dynamically defined for the action α_G. Denote the collection of Lyapunov foliations for α_G by $\mathcal{F}$. Notice that we do not know whether the leaves of those foliations are smooth manifolds. However, every foliation in $\mathcal{F}$ is integrable with $\mathcal{N}_0$ and the resulting center-Lyapunov foliation does have smooth leaves since it coincides with the center-Lyapunov foliation for $\alpha_{0,G}$.

Let $1 \le i \ne j \le n$, let $a_1, \ldots, a_m$ be different elements within the Weyl chambers in $\mathbb{G}$ and away from the Weyl chamber walls (i.e., in $\bar{A}$) such that $\mathcal{U}_{ij} = \bigcap_{k=1}^m \mathcal{W}^s_{a_k}$ (i.e., such that $\chi_{ij}(a_k) < 0$, where $\mathcal{W}^s_{a_k}$ is the stable foliation for $\alpha_0(a_k,\cdot)$ and χ_{ij} Lyapunov exponents for the action $\alpha_{0,G}$). Let $\tilde{\mathcal{F}}_{ij} = \bigcap_{k=1}^m \tilde{\mathcal{W}}^s_{a_k}$ be the corresponding Lyapunov foliations for $\tilde{\alpha}$. Denote Lyapunov foliations $\tilde{h}^{-1}\tilde{\mathcal{F}}_{ij}$ for α_G by $\mathcal{F}_{ij}$. Although we do not know whether the foliations $\mathcal{F}_{ij}$ have smooth leaves, they are simultaneously homeomorphic to the foliations $\tilde{\mathcal{F}}_{ij}$ with smooth leaves. Thus we can apply all notions developed in Section 3 to the collection $\mathcal{F}_{ij}$. We will do that from now on.

5.2 Stability of local transitivity

In [1] Brin and Pesin show that the property of local transitivity of stable and unstable foliations of a partially hyperbolic diffeomorphism persists under C^2-small perturbations. This implies the following.

Proposition 5.1. If $\tilde{\alpha}_G$ is sufficiently C^2-close to $\alpha_{0,G}$, then (1) $\tilde{\mathcal{F}}$ are LT and (2) the Lyapunov foliations of α_G are LT. □

Proof. Theorem 4.2 in [1] proves the stability of local transitivity in case of a single partially hyperbolic diffeomorphism and two foliations of high regularity (stable and unstable); the same argument applies here in the case of several smooth foliations. The statement (2) of Proposition 5.1 is an immediate corollary of the fact that $\tilde{\alpha}_G$ and α_G are conjugate via a Holder homeomorphism. ■

6 Holonomy and Canonical Projections

In this section, we will move back and forth between the phase space $X = SL(n,\mathbb{R})/\Gamma$ and its covering $SL(n,\mathbb{R})$ and use the same notation in both cases for all the invariant foliations. Since $\alpha_{0,G}$ and α_G both lift to the covering, the notions of reducible cycles and of all the collections defined in Section 3.3 such as $\mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})$ and $\mathcal{AS}_{\mathcal{F}}(\alpha_G)$ make sense and we keep here the same notation.

Remark. Notice that $SL(n,\mathbb{R})$ is not the universal cover for X but for $n \geq 3$ it is almost so, namely, the universal cover is a double cover of $SL(n,\mathbb{R})$. We could use the universal cover in the subsequent discussion, but it would make essentially no difference so we stick to the more familiar $SL(n,\mathbb{R})$. □

6.1 $\mathcal{U}$- and $\mathcal{F}$-holonomies

The foliations $\mathcal{N}_0$ and $\mathcal{U}_{ij}$ on the covering space $SL(n,\mathbb{R})$ integrate to an invariant foliation $\mathcal{W}_{ij}$ with the product structure. This foliation is also invariant for α_G. Moreover, every leaf of $\mathcal{F}_{ij}$ inside this foliation intersects every leaf of $\mathcal{N}_0$ at a unique point. Thus we can define $\mathcal{U}$-holonomy (along the leaves of $\mathcal{U}_{ij}$) and $\mathcal{F}$-holonomy (along the leaves of $\mathcal{F}_{ij}$) between different leaves of $\mathcal{N}_0$ within a leaf of $\mathcal{W}_{ij}$.

Now consider products of holonomies for different i, j pairs. In particular, for every $\mathcal{U}$-path and every $\mathcal{F}$-path with endpoints on the same leaf of $\mathcal{N}_0$, the product of these holonomies is a map of the leaf. It follows immediately from the commutation relations in $SL(n,\mathbb{R})$ that any $\mathcal{U}$-holonomy is a translation and hence depends only on the endpoints in a leaf of $\mathcal{N}_0$. Since the collection of Lyapunov foliations $\mathcal{U}_{ij}$ is LT, every translation appears as a holonomy of a leaf $\mathcal{N}_0$. Hence the group of $\mathcal{U}$-holonomies is an abelian group isomorphic to D_+ and acting simply transitively on each leaf of $\mathcal{N}_0$.

Proposition 5.1 shows that $\mathcal{F}$-foliations on X are LT. Since we consider $\mathcal{F}$-holonomies on the cover, we make use of the following fact.

Lemma 6.1. Foliations $\mathcal{F}_{ij}$, $1 \le i \ne j \le n$ on $SL(n,\mathbb{R})$ are transitive. □

Proof. We need to show that any two points x and y in $SL(n,\mathbb{R})$ can be connected by an $\mathcal{F}$-path. Since the foliations $\mathcal{F}_{ij}$ are transitive on X, it suffices to show that for a set of generators of Γ there are closed $\mathcal{F}$-paths on the compact manifold that represent those generators of the fundamental group. This is true for $\mathcal{U}$-paths. Take the corresponding $\mathcal{F}$-paths (canonical projections of the $\mathcal{U}$-paths); they may not be closed but are nearly so because the paths are of uniformly bounded length and the perturbation is small. Then, by local transitivity (Proposition 5.1), this small gap can be filled by a small $\mathcal{F}$-path. Concatenation of that path and the $\mathcal{F}$-path obtained as a projection of a $\mathcal{U}$ path representing some generator $\gamma \in \Gamma$ gives an $\mathcal{F}$-path in $SL(n,\mathbb{R})$ which represents γ. ■

Theorem 6.1. The group of $\mathcal{F}$-holonomies acts simply transitively on each leaf of $\mathcal{N}_0$. □

We postpone the proof of Theorem 6.1 to Section 7.1. Here we prove its corollary.

Corollary 6.1. The group of $\mathcal{F}$-holonomies has no compact subgroups. □

Proof. By Theorem 6.1 the group of $\mathcal{F}$-holonomies is homeomorphic to $\mathbb{R}^n$, thus it is isomorphic to a Lie group (by Montgomery and Zippin [17]). If KAN is the Iwasawa decomposition of the Lie group of $\mathcal{F}$-holonomies, then the group is homotopic to compact K and since the group of $\mathcal{F}$-holonomies is contractible, so is K. But any compact contractible topological group is trivial (see for example [8]), thus K is trivial and the group of $\mathcal{F}$-holonomies is solvable. Since a solvable simply connected Lie group cannot have compact subgroups [7, Theorem 2.3], the claim follows. ■

6.2 Correspondence between Lyapunov paths for $\alpha_{0,G}$ and for α_G

Now we define continuous maps (depending on a given foliation and a leaf) which take leaves of $\mathcal{F}_{ij}$ to leaves of $\mathcal{U}_{ij}$ and vice versa within the same center-Lyapunov leaf of $\mathcal{W}_{ij}$ on the covering space, as projections along the leaves of the neutral foliation $\mathcal{N}_0$. These maps which we call *canonical projections* will allow us further to map $\mathcal{F}$-cycles to (possibly open) $\mathcal{U}$-paths and $\mathcal{U}$-cycles to $\mathcal{F}$-paths.

In particular, since the canonical projections take pieces of the stable foliation for a regular element of one action to pieces of the stable foliation for the same element of the other action, they take *stable $\mathcal{F}$-cycles* to *stable $\mathcal{U}$-cycles* and vice versa. This is the key observation for our study of correspondences between $\mathcal{F}$-cycles and $\mathcal{U}$-cycles. However, not all $\mathcal{U}$-cycles, even contractible ones, are stable, see (4.3) in Section 4 and remarks thereof.

In this section, we relate the reducibility classes of $\mathcal{U}$-*cycles* and the reducibility classes of $\mathcal{F}$-*cycles*.

Proposition 6.1.

(1) Let $x \in SL(n,\mathbb{R})$, and $x' \in \mathcal{U}_{ij}^{x}$, $y \in \mathcal{N}_0^{x}$. Then $\mathcal{P}_{ij}^{x,y} : x' \mapsto \mathcal{F}_{ij}^{y} \cap \mathcal{N}_0^{x'}$ is a well-defined and continuous map from $\mathcal{U}_{ij}^{x}$ to $\mathcal{F}_{ij}^{y}$.

(2) Let $x \in SL(n,\mathbb{R})$, and $x' \in \mathcal{F}_{ij}^{x}$, $y \in \mathcal{N}_0^{x}$. Then $\bar{\mathcal{P}}_{ij}^{x,y} : x' \mapsto \mathcal{U}_{ij}^{y} \cap \mathcal{N}_0^{x'}$ is a well-defined and continuous map from $\mathcal{F}_{ij}^{x}$ to $\mathcal{U}_{ij}^{y}$. □

Proof. (1) We have $x' = e_{ij}(t) \cdot x$ for some $t \in \mathbb{R}$ and $y = d \cdot x$ for some $d \in D_+$. We first show that the intersection of $\mathcal{F}_{ij}^{y}$ and $\mathcal{N}_0^{x'}$ exists whenever y and x' are as above. It is immediate then, that such an intersection is unique. It follows from the fact that α_G is a conjugate of an action which is C^2-close to $\alpha_{0,G}$ that the leaves of the corresponding Lyapunov foliations are C^0 close on compact sets. But the piece of the foliation $\mathcal{U}_{ij}$ between x and x' may be rather long. So, in order to define a projection, we first make it small. To do this, choose an element in $\bar{A}$ which contracts $\mathcal{U}_{ij}$, that is, such that $\chi_{ij}(a) < 0$. Consider the $\mathcal{U}$-path along $\mathcal{U}_{ij}^{z}$ between $z := \alpha_{0,G}(a, y) = d(a, y) \cdot y$ (where $d(a, y)$ is some element in D_+) and $z' := d(a, y)d \cdot x' = e_{ij}(t') \cdot z$. For the properly chosen a, the length of this piece of $\mathcal{U}_{ij}^{z}$ leaf is sufficiently small so that the neutral leaf through z' intersects the leaf $\mathcal{F}_{ij}^{z}$ at a point w. Now define $y' := \alpha_G(a^{-1}, w)$. From the construction, this point lies both on $\mathcal{F}_{ij}^{y}$ and on $\mathcal{N}_0^{x'}$, so we may define $\mathcal{P}_{ij}^{x,y}(x') = y'$.

(2) Let $x \in SL(n, \mathbb{R})$, let $1 \le i \ne j \le n$ and let $x' \in \mathcal{F}_{ij}^{x}$. Let $y \in \mathcal{N}_0^x$, $y = d \cdot x$ for some $d \in D_+$. Choose an element $a \in \bar{A}$ which contracts $\mathcal{F}_{ij}$. Translate the leaf $\mathcal{F}_{ij}^x$ by this element and obtain points $z := \alpha_G(a, x)$ and $z' := \alpha_G(a, x') \in \mathcal{F}_{ij}^z$. Now if a was chosen properly, the piece between z and z' is small enough so that $\mathcal{N}_0^{z'}$ intersects $\mathcal{U}_{ij}^z$ at some point w. Since $z = \alpha_G(a, x) \in \mathcal{N}_0^y$, $z = d_1 \cdot y$ for some $d_1 \in D_+$. Therefore, the point $y' := d_1^{-1} \cdot w$ lies both on $\mathcal{U}_{ij}^y$ and on $\mathcal{N}_0^{x'}$, so we may define $\bar{\mathcal{P}}_{ij}^{x,y}(x') := y'$. ■

Maps $\mathcal{P}_{ij}^{x,y}$ and $\bar{\mathcal{P}}_{ij}^{x,y}$ we call *canonical projections.* It is clear that:

$$\bar{\mathcal{P}}_{ij}^{y,x}(\mathcal{P}_{ij}^{x,y}(x')) = x' \quad \text{and} \quad \mathcal{P}_{ij}^{y,x}(\bar{\mathcal{P}}_{ij}^{x,y}(x')) = x'.$$

Definition 6.1. Let $\mathfrak{c} : x_1, \ldots, x_{m-1}, x_m = x_1$ be a $\mathcal{U}$-cycle with initial point x_1. Then the *canonical projection* of this path at a point $y := d \cdot x_1$, $d \in D_+$ is an $\mathcal{F}$-path $\mathcal{P}^{x_1,y}(\mathfrak{c}) : y_1 = y, y_2, \ldots, y_m$ such that for each $k \in \{1, \ldots, m-1\}$ we have $y_{k+1} := \mathcal{P}_{i(k)j(k)}^{x_k,y_k}(x_{k+1})$. □

The projected path $\mathcal{P}^{x_1,y}(\mathfrak{c})$ need not be closed, but since the projection is along the leaves of the neutral foliation $\mathcal{N}_0$ we have that $y_m \in \mathcal{N}_0^{y_1}$, that is, $y_m = d^{x,y}(\mathfrak{c}) \cdot y_1$ for some $d^{x,y}(\mathfrak{c}) \in D_+$. The map $(x, y, \mathfrak{c}) \to d^{x,y}(\mathfrak{c})$ is continuous due to continuity of projections.

Similarly, define the reverse projection $\bar{\mathcal{P}}^{x_1,y}(\mathfrak{c})$ of an $\mathcal{F}$-cycle $\mathfrak{c}$ with an initial point x_1 and to a (possibly open) $\mathcal{U}$-path starting at y, using canonical projections $\bar{\mathcal{P}}_{ij}^{x,y}$. The difference between the endpoints along the leaves of $\mathcal{N}_0$ will be denoted by $\bar{d}^{x,y}(\mathfrak{c})$. The following lemma is one of the main links between unperturbed (algebraic) and the perturbed (nonalgebraic) setting.

Lemma 6.2. Canonical projections have the following properties:

(1) $\mathcal{P}^{x,y} : \mathcal{AS}_{\mathcal{U}}^{s}(\alpha_{0,G})^x \to \mathcal{AS}_{\mathcal{F}}^{s}(\alpha_G)$ and $\bar{\mathcal{P}}^{x,y} : \mathcal{AS}_{\mathcal{F}}^{s}(\alpha_G)^x \to \mathcal{AS}_{\mathcal{U}}^{s}(\alpha_{0,G})$.

(2) $\mathcal{P}^{x,y} : \mathcal{AS}_{\mathcal{U}}^{rs}(\alpha_{0,G})^x \to \mathcal{AS}_{\mathcal{F}}^{rs}(\alpha_G)$ and $\bar{\mathcal{P}}^{x,y} : \mathcal{AS}_{\mathcal{F}}^{rs}(\alpha_G)^x \to \mathcal{AS}_{\mathcal{U}}^{rs}(\alpha_{0,G})$.

(3) $\mathcal{P}^{x,y} : \mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})^x \to \mathcal{AS}_{\mathcal{F}}(\alpha_G)$ and $\bar{\mathcal{P}}^{x,y} : \mathcal{AS}_{\mathcal{F}}(\alpha_G)^x \to \mathcal{AS}_{\mathcal{U}}(\alpha_{0,G})$.

(4) Canonical projections $\mathcal{P}^{x,y}$ map $\alpha_{0,G}$-reducible $\mathcal{U}$-cycles in $\mathcal{C}^x(\mathcal{U})$ to α_G-reducible $\mathcal{F}$-cycles in $\mathcal{C}^y(\mathcal{F})$ and conversely $\bar{\mathcal{P}}^{x,y}$ map α_G-reducible $\mathcal{F}$-cycles in $\mathcal{C}^x(\mathcal{F})$ to $\alpha_{0,G}$-reducible $\mathcal{U}$-cycles in $\mathcal{C}^y(\mathcal{U})$. □

Proof. Lemma 6.2 is an immediate consequence of the Definition 6.1 of canonical projections, that is, the fact that any canonical image of a stable leaf is a stable leaf, and the continuity of canonical projections. ■

Remark. Since reducible cycles project to reducible *cycles*, it follows immediately that canonical projections of any two cycles in the same reducibility class have the same endpoints on the leaf of $\mathcal{N}_0$, that is, the maps $d^{x,x}$ and $\bar{d}^{x,x}$ do not depend on the representative from the given reducibility class. □

6.3 Canonical projections of $\mathcal{U}$-cycles

Lemma 6.3. Let $\mathfrak{u}$ be a $\mathcal{U}$-cycle.

(1) If $\mathfrak{u}$ is contractible, the projection of $\mathfrak{u}$ is an $\mathcal{F}$-cycle.

(2) If $\mathfrak{u}$ is not contractible, its projection is an $\mathcal{F}$-path such that the distance between its endpoints is small of the order of the smallness of the perturbation α_G. □

Proof. (1) The claim for contractible $\mathcal{U}$-cycles is a consequence of the discussion preceding Proposition 4.2 and of Lemma 6.2. Namely every contractible $\mathcal{U}$-cycle in $SL(n,\mathbb{R})$ is represented by a relation in the group $SL(n,\mathbb{R})$ which is $\alpha_{0,G}$-reducible due to the Steinberg theorem and Lemma 4.1. Now since $\alpha_{0,G}$-reducible $\mathcal{U}$-cycles project to α_G-reducible $\mathcal{F}$-cycles, the claim follows.

(2) If a $\mathcal{U}$-cycle $\mathfrak{u}$ in $SL(n,\mathbb{R})$ is not contractible, then it is in the same homotopy class, thus in the same reducibility class, as a cycle generated by the relation $\{-1,-1\}=id$. Such a cycle is given by the relation $(e_{12}(-1)e_{21}(1)e_{12}(-2)e_{21}(1)e_{12}(-1))^2=1$ so it is symmetric, so a $\mathcal{U}$-cycle $\mathfrak{u}_0$ generated by this relation projects in the same way as its inverse. So even though $\mathfrak{u}_0^2$ projects to a closed $\mathcal{F}$ path, the cycle $\mathfrak{u}_0$ may project to an open $\mathcal{F}$-path with opening $d\in D_+$. It is still true, however, that due to the smallness of the perturbation d must be small for this $\mathfrak{u}_0$ and thus for any other $\mathcal{U}$-cycle $\mathfrak{u}$ which is in the same reducibility class as $\mathfrak{u}$. ■

6.4 Canonical projections of $\mathcal{F}$-cycles

Let now C^x denote the collection of contractible $\mathcal{F}$-cycles starting at x in $SL(n,\mathbb{R})$ and define the following subset of D_+:

$$D(x) := \{\bar{d}^{x,x}(\mathfrak{c}) : \mathfrak{c}\in C^x\}$$

where $\bar{d}^{x,x}(\mathfrak{c})$ is defined in Section 6.2 via endpoints of $\mathcal{U}$ paths which are obtained as canonical projections $\mathcal{P}^{x,x}(\mathfrak{c})$.

Since for $\mathfrak{c} \in C^x$ the element $\bar{d}^{x,x}(\mathfrak{c}) \in D_+$ does not depend on the initial point of the projection, the set $D(x)$ is a subgroup of D_+ and we will denote $\bar{d}^{x,x}(\mathfrak{c})$ simply by $\bar{d}(\mathfrak{c})$ whenever it is clear what the initial point x is.

Lemma 6.4. For all x, $D(x)$ is the same subgroup of D_+, denoted by D. □

Proof. Let $\mathfrak{c}$ be in C^x and such that $\bar{d}(\mathfrak{c}) \neq 1$. Let $z \in \mathcal{F}_{ij}^x$ and consider the closed path beginning at z which is obtained from $\mathfrak{c}$ by adding at point x the piece of the leaf of $\mathcal{F}_{ij}$ from z to x at the beginning and in the opposite direction at the end of $\mathfrak{c}$. Call this new path $\bar{\mathfrak{c}}$.

The endpoints of any two projections of a two point path contained in a leaf of some $\mathcal{F}_{ij}$ are obtained by the action of the *same* element of the full WCF. If $\bar{d}(\mathfrak{c}) = \operatorname{diag}(d_1, \dots, d_n)$, then direct matrix multiplication shows that $e_{ij}(t) \cdot \bar{d}(\mathfrak{c}) \cdot e_{ij}(s_{ij}(t)) \in D_+$ implies for every pair $i, j, i \neq j$, $d_i s_{ij}(t) = -d_j t$. Moreover, in this case, we have $e_{ij}(t) \cdot \bar{d}(\mathfrak{c}) \cdot e_{ij}(s_{ij}(t)) = \bar{d}(\mathfrak{c})$, so by moving along the leaves of foliations $\mathcal{F}_{ij}$ the $\mathcal{N}_0$ distance between the endpoints of any projection to $\mathcal{U}$-paths does not change. In particular, $D(x)$ does not change along the leaves of foliations $\mathcal{F}_{ij}$. By Lemma 6.1, foliations $\mathcal{F}_{ij}$, $1 \leq i, j \leq n$ constitute a transitive system on $SL(n, \mathbb{R})$. This implies that $D(x)$ is constant everywhere. ■

Lemma 6.5. The group D is discrete. □

Proof. For any $d \in D_+$, there is a canonical (unique) way of representing it as a product of elementary diagonal matrices $h_{ij}(t)$. Namely,

$$d = h_{12}(t_1) \cdot h_{23}(t_2) \dots h_{n-1,n}(t_{n-1})$$

and for each $i \in \{1, \dots, n-1\}$ and any $t \in \mathbb{R}^*$, $h_{i,i+1}(t)$ is defined as in the Steinberg theorem. For any $x \in SL(n, \mathbb{R})$, the above expression for d defines a $\mathcal{U}$-path which consists of $6(n-1)$ arcs along leaves of foliations $\mathcal{U}_{ij}$ for various (i, j). Call such a path a *standard* $\mathcal{U}$-path. So, to any $d \in D$ and to any point x, there corresponds a standard $\mathcal{U}$-path denoted by $\mathfrak{u}_{d,x}$ which connects x and dx.

Assume that the connected component of the identity D_0 in the group D is nontrivial. Pick $d \in D_0$ of norm 1. Let $\mathfrak{c}$ be an $\mathcal{F}$-cycle in C^x such that $\bar{d}(\mathfrak{c}) = d$. Now to d of norm 1, there corresponds a standard $\mathcal{U}$-path $\mathfrak{u}_{d,x}$ consisting of a fixed number of arcs (at most $6(n-1)$ arcs) and moreover, the lengths of these arcs are uniformly bounded since d is of norm 1. Let $\tilde{\mathfrak{u}}$ be the closed $\mathcal{U}$ path which consists of the standard $\mathcal{U}$-path $\mathfrak{u}_{d,x}$ and the projection $\mathfrak{u} = \bar{\mathcal{P}}^{x,x}(\mathfrak{c})$.

By Lemma 6.3, the projection $\mathcal{P}^{x,x}(\tilde{\mathfrak{u}})$ is a closed $\mathcal{F}$-path if $\tilde{\mathfrak{u}}$ is contractible, and since $\mathfrak{c}$ is a closed $\mathcal{F}$-path, it follows that the projection of the standard path $\mathfrak{c}_d := \mathcal{P}^{x,x}(\mathfrak{u}_{d,x})$ is also a closed $\mathcal{F}$-path at x.

Since the length of arcs in $\mathfrak{u}_{d,x}$ is uniformly bounded, each of the links of $\mathcal{P}^{x,x}(\mathfrak{u}_{d,x})$ is close (the order of the smallness of the perturbation) to the corresponding link of $\mathfrak{u}_{d,x}$ and for any d the number of links of $\mathcal{P}^{x,x}(\mathfrak{u}_{d,x})$ is at most $6(n-1)$. Hence the endpoint dx is close to x, that is, d has to be small which is in contradiction with d being of norm 1.

If $\tilde{\mathfrak{u}}$ is not contractible, then one only needs to double it in order to get a contractible path, so the argument above applies in the same way, that is, d is obtained by projecting a $\mathcal{U}$-cycle with a fixed number of links of bounded length, thus has to be small, which again contradicts d being of norm 1.

Thus, $D_0 = \{id\}$ and D is discrete. ■

Corollary 6.2. Every $\mathcal{F}$ path of sufficiently small diameter is α_G-reducible. □

Lemma 6.6. The group D is trivial. □

Proof. Without loss of generality (taking the double if necessary), if D is not trivial we may suppose that there is a (very long) contractible $\mathcal{F}$-cycle $\mathfrak{c}$ with base point x whose canonical projection is not closed. Let c_s, $0 \le s \le 1$ be a homotopy fixing x between $\mathfrak{c} = c_0$ and the trivial cycle c_1.

For any $\epsilon > 0$, one can find M such that we can construct a finite sequence of $\mathcal{F}$ cycles $\mathfrak{c}^0 = \mathfrak{c} = c_0, \mathfrak{c}_1, \ldots, \mathfrak{c}^M = c_1$ such that for $k = 0, 1, \ldots, M$ $\mathfrak{c}_k$ is C^0, ϵ close to $c_{k/M}$ as a parametrized path, that is, a map $[0, 1] \to X$. If M is chosen large enough, then C^0 distance between $\mathfrak{c}^k$ and $\mathfrak{c}^{k+1}$ will be $< 2\epsilon$ for each k, $0 \le k \le M-1$.

If ϵ is chosen small enough, this implies that the cycles $\mathfrak{c}^k$ and $\mathfrak{c}^{k+1}$ are α_G equivalent. For, the cycle $\mathfrak{c}^{k+1} * (c^k)^{-1}$ is conjugate to a composition of cycles of diameter $< 4\epsilon$. Each of those cycles is α_G-reducible by Corollary 6.2. Hence, the original cycle $\mathfrak{c}$ is equivalent to the trivial cycle and hence is α_G-reducible. This implies by Lemma 6.2 that its canonical projection is a $\mathcal{U}$-cycle, a contradiction. ■

Corollary 6.3. Every projection of every $\mathcal{F}$-cycle on $SL(n, \mathbb{R})$ is a $\mathcal{U}$-cycle. □

Proposition 6.2. If $\mathfrak{c}$ is a contractible $\mathcal{F}$-cycle, then $\mathfrak{c}$ or $\mathfrak{c} * \mathfrak{c}$ is α_G-reducible. □

Proof. Let $\mathfrak{c}$ be a contractible $\mathcal{F}$-cycle in X. Then, it lifts to a contractible $\mathcal{F}$-cycle $\bar{\mathfrak{c}}$ in $SL(n, \mathbb{R})$ starting at some x. Since $\bar{d}^{x,x}(\bar{\mathfrak{c}}) = id$, the $\mathcal{U}$-path $\mathfrak{u} = \bar{P}^{x,x}(\bar{\mathfrak{c}})$ is a $\mathcal{U}$-cycle on the cover and therefore it is reducible, or reducible if doubled (Lemma 6.3), so by Lemma 6.2, the same holds for $\bar{\mathfrak{c}}$ and thus the same holds for $\mathfrak{c}$. ■

7 Proof of Theorems 1.2 and 6.1

7.1 Simple transitivity of $\mathcal{F}$-holonomy: proof of Theorem 6.1

The $\mathcal{F}$-holonomy group acts transitively due to the transitivity of the system of foliations $\mathcal{F}_{ij}$. Now let x, y be on the same leaf of $\mathcal{N}_0$ and consider two different $\mathcal{F}$-paths $\mathfrak{p}_1$ and $\mathfrak{p}_2$ connecting x and y. Then the $\mathcal{F}$-path $\mathfrak{c}'$ corresponding to the cycle $\mathfrak{c} := \mathfrak{p}_1 * \mathfrak{p}_2^{-1}$ starting at any other point on the leaf of $\mathcal{N}_0$ is closed. If $\mathfrak{c}$ is α_G-reducible, this is due to the fact that (the α_G-action induced) projections along the leaves of $\mathcal{N}_0$ from leaves of $\mathcal{F}$ foliations to leaves of $\mathcal{F}$-foliations preserve reducibility classes, so $\mathfrak{c}'$ is also reducible, so it is an $\mathcal{F}$-cycle. If $\mathfrak{c}$ is not in the trivial reducibility class, then the map taking $\mathfrak{c}$ to the distance between the endpoints of $\mathfrak{c}'$ at some different point on the leaf of $\mathcal{N}_0$ gives a homomorphism from the group of α_G-reducibility classes into D_+ and every such homomorphism is trivial by Proposition 6.2 and the fact that there are no nontrivial homomorphisms from Γ into D_+ ([13, Theorem (4), Chapter 1]). Hence, the corresponding holonomy map is the identity and the holonomy maps corresponding to $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are the same.

7.2 Cocycle rigidity for a C^2-small perturbation of a generic restriction: proof of Theorem 1.2

Let $\tilde{\alpha}_G$ be a C^2-small perturbation of a generic restriction $\alpha_{0,G}$. Let $\tilde{\mathcal{F}}$ denote the collection of Lyapunov foliations of $\tilde{\alpha}_G$. Let α_G be the action obtained by conjugating $\tilde{\alpha}_G$ by the Hirsch–Pugh–Shub homeomorphism as described in Section 2.1. As before, $\mathcal{F}$ denotes the collection of Lyapunov foliations of α_G.

By Proposition 6.2, the assumption 1 in Proposition 3.2 is satisfied for $\mathcal{F}$-foliations of α_G. Since this property is preserved by topological conjugacy, Lyapunov foliations for $\tilde{\alpha}_G$, that is, $\tilde{\mathcal{F}}$-foliations, also satisfy the first assumption in Proposition 3.2. Now Theorem 1.2 is an immediate consequence of Proposition 3.2 and the fact that due to the higher rank assumption any homomorphism from $\pi_1(X)$ into $\mathbb{R}^l$ is trivial [13, Theorem (4), Chapter 1].

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