TWO PROBLEMS IN MEASURE RIGIDITY

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1. Existence and arithmeticity of maximal rank hyperbolic measures

Theorem 1. [4] *Let $k \geq 2$, α be a $\mathcal{C}^{1+\epsilon}$, $\epsilon > 0$ action of $\mathbb{Z}^k$ on a $k+1$-dimensional manifold , μ an ergodic invariant measure of α with no proportional Lyapunov exponents and at least one element of α has positive entropy.*

Then μ is absolutely continuous.

The only known model for such action is the algebraic Cartan action on the torus $\mathbb{T}^{k+1}$, i.e. the action by hyperbolic maps with real eigenvalues. All known examples are differentiably conjugate to a Cartan action on an invariant open set. Notice however that there are many manifolds which can carry such actions even if one requires topological transitivity in addition. Those manifolds are constructed by blowing up periodic orbits of a Cartan action and either glueing in projective spaces (a σ-process) or identifying boundary spheres of different holes.

Problem 1. *What compact manifolds carry actions satisfying assumptions of Theorem 1?*

The answer may be different for real-analytic actions where certain restrictions are plausible and smooth (C^∞) actions which are likely to exist on any compact manifold. The key case is that of the ball $\mathbb{D}^{k+1}$. In fact existence of an action on the ball which is sufficiently "flat" at the boundary would imply existence on any compact manifold as in [1].

The most interesting problem concerns certain arithmetic structure present in such actions. It is motivated by the following result for the torus.

Theorem 2. [3, 5] *Let α be a $\mathcal{C}^{1+\epsilon}, \epsilon > 0$, $\mathbb{Z}^k$ action on $\mathbb{T}^{k+1}$ Cartan homotopy data i.e. each element is homotopic to the corresponding element of a linear Cartan action α_0. Then*

- *The set $\mathcal{M}$ consists of a single measure μ.*
- *The measure μ is absolutely continuous.*
- *The semi-conjugacy h is bijective on a set of full measure and thus effects a measurable isomorphism between (α, μ) and (α_0, λ).*
- *The semi-conjugacy is differentiable along almost every leaf of each Lyapunov foliation.*

Problem 2. *What are possible values of entropy for elements of an action α satisfying assumptions of Theorem 1?*

The following conjecture represents a cautiously optimistic view of the situation.

Conjecture 1. *The entropy values are algebraic integers of degree at most $k+1$.*

2. Higher rank global symplectic rigidity?

Theorem 2 represents the first case of global measure rigidity on the torus. Its proof is based on ideas developed in [6].

Proofs of measure rigidity of linear actions other than linear Cartan or more general *totally non-symplectic* requires different methods; see [2].

As a representative example for a possible global rigidity result consider a linear action α_0 of a maximal abelian subgroup of $SP(4,\mathbb{Z})$ on $\mathbb{T}^4$ diagonalizable over $\mathbb{R}$. Let α be a $\mathbb{Z}^2$ action whose elements are homotopic to the corresponding elements of α_0 and let μ be an α-invariant Borel probability measure such that $h_*\mu = \lambda$ where h is the semiconjugacy between α and α_0 as before.

Problem 3. *Show that μ is absolutely continuous and that $\mu = \omega \wedge \omega$ where ω is a Lebesgue measurable exterior 2-form closed in a properly defined sense.*

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