RANDOM PERTURBATIONS OF TRANSFORMATIONS OF AN INTERVAL

By

A. KATOK AND Y. KIFER

Abstract. Let μ^ε be invariant measures of the Markov chains x_n^ε which are small random perturbations of an endomorphism f of the interval $[0,1]$ satisfying the conditions of Misiurewicz [6]. It is shown here that in the ergodic case μ^ε converges as $\varepsilon \to 0$ to the smooth f-invariant measure which exists according to [6]. This result exhibits the first example of stability with respect to random perturbations while stability with respect to deterministic perturbations does not take place.

0. Introduction

Let f be a C^3 map of the interval $I = [0,1]$ into itself. Consider a family of probability measures $Q^\varepsilon(x, dy)$ on I given for every $x \in I$ and $\varepsilon > 0$ small enough. Define the Markov chain x_n^ε, $n = 0, 1, \ldots$ in the following way: if $x_n^\varepsilon = x$ then x_{n+1}^ε has distribution $Q^\varepsilon(fx, dy)$. The Markov chains x_n^ε are called small random perturbations of the transformation f if for each continuous function h on I,

$$\lim_{\varepsilon \to 0} \sup_{x \in I} \left| \int_I Q^\varepsilon(x, dy) h(y) - h(x) \right| = 0, \tag{0.1}$$

where we shall consider the interval I both with the identification of endpoints and without it.

We shall say that a probability measure μ^ε on I is an invariant measure of the Markov chain x_n^ε if for any Borel set $\Gamma \subset I$,

$$\int_I \mu^\varepsilon(dx) P^\varepsilon(x, \Gamma) = \mu^\varepsilon(\Gamma) \tag{0.2}$$

where

$$P^\varepsilon(x, \Gamma) = Q^\varepsilon(fx, \Gamma). \tag{0.3}$$

It follows easily from (0.1)–(0.3) (see [5]) that if μ^ε are invariant measures of small random perturbations x_n^ε of the transformation f and

$$\mu^{\varepsilon_i} \to \mu \text{ in the weak sense } (\mu^{\varepsilon_i} \xrightarrow{w} \mu) \text{ for some subsequence } \varepsilon_i \to 0 \tag{0.4}$$

then μ is an invariant measure of the map f, i.e. for any Borel set $\Gamma \subset I$,

$$\mu(f^{-1}\Gamma) = \mu(\Gamma). \tag{0.5}$$

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 47 (1986)

If the limit measure μ in (0.4) is the same for any subsequence $\varepsilon_i \to 0$ then it is said to be stable with respect to random perturbations.

We shall study here the limit behaviour of the measures μ^ε in the case of maps f satisfying the conditions of Misiurewicz [6], i.e. having non-positive Schwarzian derivative, no sinks and trajectories of critical points stay away from critical points. These mappings f possess absolutely continuous invariant measures (see [6]). For the most widely considered one-parameter family of maps $x \to 4\lambda x(1-x)$ the conditions of [6] are satisfied for a set of parameters λ having cardinality of the continuum.

We shall prove in this paper that if the transformation $f_\lambda(x) = 4\lambda x(1-x)$ satisfies the conditions of [6] and it is ergodic with respect to its absolutely continuous invariant measure μ_f then the limit measure in (0.4) is always μ_f for a wide class of random perturbations. The proof follows the lines of [5] with modifications due to the fact that our transformation is not uniformly hyperbolic. The exact conditions on perturbations will be discussed in the next section. Still, we remark here that the following is a particular case of our model. Let $\lambda_1, \lambda_2, \ldots$ be independent random variables with the same distribution having a smooth density concentrated on $[-1,1]$. Suppose that $\bar\lambda$ is a parameter such that $\frac{1}{2} < \bar\lambda < 1$ and the map $x \to 4\bar\lambda x(1-x)$ satisfies the conditions of [6] mentioned above. Then for $\varepsilon < 1 - \bar\lambda$ the composition of independent random transformations $f_{\bar\lambda+\varepsilon\lambda_1}, f_{\bar\lambda+\varepsilon\lambda_2}, \ldots$ generates a Markov chain

$$x_n^\varepsilon = f_{\bar\lambda+\varepsilon\lambda_n} \circ \cdots \circ f_{\bar\lambda+\varepsilon\lambda_1} x$$

which belongs to the class of random perturbations of $f_{\bar\lambda}$ satisfying our conditions. The case of $\bar\lambda = 1$ must be treated in a slightly modified way.

The stability of measures μ_f with respect to random perturbations is especially interesting in view of the fact that in general there is no stability with respect to deterministic perturbations in this case. Indeed, consider the family of transformations $f_\lambda(x) = 4\lambda x(1-x)$ with λ close to 1. Clearly, $f_1(x)$ satisfies the conditions of [6] and it has absolutely continuous invariant measure with the density

$$p(x) = \frac{1}{\pi}(x(1-x))^{-1/2}$$

with respect to the Lebesgue measure on $[0,1]$. Define $n_\lambda = \min\{n > 1 : f_\lambda^n(\frac{1}{2}) \geqq \frac{1}{2}\}$. Since $f_1^n(\frac{1}{2}) = 0$ for all $n > 1$ then if $f_\lambda^{n_\lambda}(\frac{1}{2}) > \frac{1}{2}$ by the continuity one can find $\beta(\lambda)$ such that $1 > \beta(\lambda) > \lambda$ and $f_{\beta(\lambda)}^{n_\lambda}(\frac{1}{2}) = \frac{1}{2}$. Therefore $\frac{1}{2}$ is a periodic point of $f_{\beta(\lambda)}$ and the corresponding periodic orbit is an attracting one since $f'_\lambda(\frac{1}{2}) = 0$ for any λ.

Hence we have found a sequence $\lambda_k \uparrow 1$ such that any f_{λ_k} has an attracting periodic orbit containing $\frac{1}{2}$ and only one point of this orbit can be to the right of $\frac{1}{2}$. The invariant measure ν_{λ_k} supported by this periodic orbit is stable with respect to random perturbations since the complement of the basin of this periodic orbit has zero Lebesgue measure (see [3], Proposition II. 5.7). On the other hand, these measures ν_{λ_k} do not converge as $\lambda_k \to 1$ to the smooth invariant measure of f_1 since these periodic orbits have only one point to the right of $\frac{1}{2}$ and so all weak limits of the corresponding invariant measures have support in the interval $[0, \frac{1}{2}]$.

One can assign to any random perturbation some point of the coordinate plane where the x-coordinate measures the deterministic part of the perturbation and the y-coordinate measures the random part of the perturbation. In this setting our results can be interpreted in the way that when perturbations approach zero along any straight line passing through zero except the x-axis then invariant measures of perturbations weakly converge to the corresponding absolutely continuous measure μ_f. On the other hand, if perturbations approach zero along a curve which is close enough to the x-axis then the convergence may not take place.

Since computer experiments are always subject to random errors our approach may explain why computations show whole subintervals of parameters λ without attracting orbits of the maps f_λ, i.e. exhibiting the behaviour as if the characteristic exponent of f_λ is positive.

We shall prove our results for the one parameter family of transformations $f_\lambda(x) = 4\lambda x(1-x)$. The reader can easily check that all arguments go on for more general one parameter families of transformations of an interval of the type considered in §7 of [6]. These should be a family $\{f_\alpha\}_{\alpha\in[0,1]}$ of unimodal maps of an interval with negative Schwartzian derivative having no sinks and the critical point should not belong to the closure of the set of its images under f_α^i, $i = 1, 2, \ldots$.

Since this paper was first written we have received the manuscrint of Collet [2] which has another model of random perturbations of transformations of an interval which is a partial case of our scheme. Actually, the same model appeared earlier in [1] for the case of random perturbations of Lasota–Yorke type expanding transformations. In this model one makes the rather strong assumption that the density $q_x^\varepsilon(y)$ of the distribution $Q^\varepsilon(x, dy)$ depends only on the difference $y - x$, i.e. it is translation invariant. Apart from the obvious reason that this type of perturbation cannot be considered, in general, in the case of dynamical systems on manifolds, the model in question does not include also the case of random perturbations of the parameter λ in f_λ which we have discussed above. The translation invariance of $q_x^\varepsilon(y) = q^\varepsilon(y-x)$ enables one to consider the Perron–Frobenius operator corresponding to the transition probability of the Markov chain x_n^ε which turns out to be in this case just the convolution of q^ε with the Perron–Frobenius operator of the map f_λ itself. This simplifies the proof enormously since instead of studying the dynamics of the Markov chain x_n^ε which is necessary in our case, it suffices to establish some properties of the Perron–Frobenius operator similarly to the proof of existence of a smooth invariant measure for a map f_λ.

It seems that our proof with some additional work can be carried out for the maps satisfying Jacobson's conditions (see [4]) which hold for the set of parameters λ of positive Lebesgue measure. On the other hand, it is not clear whether our theorem has any connection to the fact that the parameters λ for which f_λ satisfies Misiurewicz conditions are points of Lebesgue density one for the set of parameters producing maps with Jacobson's conditions and so having absolutely continuous invariant measures (see [4]).

The general ideas of this paper were discussed by the authors in July 1983 at the University of Maryland when both authors were supported by N.S.F. Grant MCS82–04024. The preliminary version was revised in Spring of 1984 while both authors were visiting the Mathematical Sciences Research Institute at Berkeley.

1. Assumptions and main results

Consider a family of non-negative functions

$$\{r_x(\xi),\ x \in I = [0,1],\ \xi \in R^1 = (-\infty, \infty)\}$$

satisfying the following Assumption A:

(i) $\int_{R^1} r_x(\xi) d\xi = 1$.

(ii) There exists C_0, $\alpha_1 > 0$ independent of x such that

$$r_x(\xi) \leqq C_0 e^{-\alpha_1 |\xi|}. \tag{1.1}$$

(iii) There exists $C_1 > 0$ such that if

$$V_x^+ \equiv \{\xi \colon r_x(\xi) > 0\}$$

then for any $x, y \in I$, $\xi \in V_x^+$ and $\eta \in V_y^+$ one has

$$|r_x(\xi) - r_y(\eta)| \leqq C_1(|x-y| + |\xi - \eta|). \tag{1.2}$$

(iv) There exists $C_2 > 0$ independent of x such that if

$\partial V_x^+(\delta)$ is a δ-neighborhood in R^1 of the boundary ∂V_x^+ of V_x^+

then

$$\int_{\partial V_x^+(\delta)} r_x(\xi) d\xi \leqq C_2 \delta \tag{1.3}$$

for any $\delta > 0$, and if $r_x(\xi) \neq 0$ but $r_y(\eta) = 0$ then

$$\xi, \eta \in \partial V_x^+(C_2(|x-y| + |\xi - \eta|)) \tag{1.4}$$

provided x and y are close enough to each other.

In fact, for the proof of Proposition 3.1 which we shall give in the Appendix, we shall need the following condition which is stronger than (1.3). It seems that this condition is not necessary for the truth of Proposition 3.1 so we add it with some reservations.

(v) For each $x \in I$ the number of points of discontinuity of $r_x(\xi)$ in ξ is bounded by a number N independent of x and on each interval of continuity $r_x(\xi)$ is Lipschitz continuous in ξ with the constant C_1.

The conditions (iii) and (iv) enable us to consider the functions $r_x(\xi)$ with compact supports V_x^+ having discontinuity on ∂V_x^+. Nevertheless the condition (iv) requires that the domains V_x^+ depend on x in a continuous manner. In particular, $r_x(\xi)$ can be the density of the uniform distribution on some interval depending on x, or in the simplest case, independent of x, say $(-1, 1)$.

For the sake of simplicity we shall consider only quadratic maps

$$f_\lambda(x) = 4\lambda x(1-x). \tag{1.5}$$

Define

$$\mathcal{T}_\lambda \equiv \overline{\bigcup_{n \geqq 1} f_\lambda^n(\tfrac{1}{2})}. \tag{1.6}$$

Throughout this paper we assume that the probability distributions $Q^\varepsilon(x, dy)$ have densities $q_x^\varepsilon(y)$ with respect to the Lebesgue measure, i.e., for any Borel set $\Gamma \subset I$,

$$Q^\varepsilon(x, \Gamma) = \int_\Gamma q_x^\varepsilon(y)dy. \tag{1.7}$$

We suppose also that for some positive $\alpha_2 < 1$ and any $x \in [\varepsilon^{\alpha_2}, 1-\varepsilon^{\alpha_2}]$,

$$q_x^\varepsilon(y) \leqq (1+\varepsilon^{\alpha_2})\varepsilon^{-1} r_x\left(\frac{y-x}{\varepsilon}\right) \qquad \text{provided } |y-x| \leqq \varepsilon^{\alpha_2}. \tag{1.8}$$

The assumptions on $q_x^\varepsilon(y)$ for $|x| < \varepsilon^{\alpha_2}$ or $|1-x| < \varepsilon^{\alpha_2}$ will depend on the type of boundary conditions we shall accept.

For the sake of simplicity we shall consider in this paper only periodic boundary conditions, i.e. we shall identify the end points 0 and 1. This means that we assume (1.8) to be true for any $x \in I$ and

$$q_x^\varepsilon(y) \leqq (1+\varepsilon^{\alpha_2})\varepsilon^{-1} r_x\left(\frac{y-x\pm 1}{\varepsilon}\right) \qquad \text{provided } |y-x\pm 1| \leqq \varepsilon^{\alpha_2}, \tag{1.9}$$

where plus and minus in the last inequality corresponds to plus and minus in the first one, respectively.

For $x, y \in I$ define

$$\operatorname{dist}(x,y) = \min(|y-x|, |y-x+1|, |y-x-1|). \tag{1.10}$$

We assume also that

$$q_x^\varepsilon(y) \leqq \exp\left(-\frac{\alpha_2}{\varepsilon}\operatorname{dist}(x,y)\right) \tag{1.11}$$

if $\operatorname{dist}(x,y) > \varepsilon^{\alpha_2}$ and $\varepsilon > 0$ is small enough.

Remark 1.1. Another boundary condition which can be treated by our method and for which the Theorem below remains true is the reflection condition in the endpoints 0 and 1. This means that (1.8) remains the same for either $x \in [\varepsilon^{\alpha_2}, 1-\varepsilon^{\alpha_2}]$ or $x < \varepsilon^{\alpha_2}$ and $y \geqq x$, or $x > 1-\varepsilon^{\alpha_2}$ and $y \leqq x$. But if $x < \varepsilon^{\alpha_2}$ and $y < x$ then one assumes

$$\varepsilon q_x^\varepsilon(y)\left(r_x\left(\frac{y-x}{\varepsilon}\right)+r_x\left(-\frac{(x+y)}{\varepsilon}\right)\right)^{-1} \leqq 1+\varepsilon^{\alpha} \tag{1.12}$$

and if $x > 1-\varepsilon^{\alpha_2}$ and $y > x$ then

$$\varepsilon q_x^\varepsilon(y)\left(r_x\left(\frac{y-x}{\varepsilon}\right)+r_x\left(\frac{2-(x+y)}{\varepsilon}\right)\right)^{-1} \leqq 1+\varepsilon^{\alpha_2}. \tag{1.13}$$

In this case (1.11) should be replaced by

$$q_x^\varepsilon(y) \leqq \exp\left(-\frac{\alpha_2}{\varepsilon}|y-x|\right) \tag{1.14}$$

if $|y-x| > \varepsilon^{\alpha_2}$ and $\varepsilon > 0$ is small enough.

For the case of quadratic maps of the form (1.8) Misiurewicz [6] proved that if f_λ has no stable periodic orbit and

$$\tfrac{1}{2} \notin \mathcal{T}_\lambda \tag{1.15}$$

with $\mathcal{T}_\lambda$ defined by (1.16) then f_λ has exactly one absolutely continuous invariant measure μ_{f_λ} which is ergodic.

Now we can state our main result

Theorem. *Suppose that* (1.7)–(1.9), (1.11) *and Assumption A are satisfied. Assume the transformation f_λ has the form* (1.5), *has no stable periodic orbit and* (1.15) *is true. If μ^ε is an invariant measure of the Markov chain x_n^ε defined in the Introduction then*

$$\mu^\varepsilon \xrightarrow{w} \mu_{f_\lambda} \qquad \text{as } \varepsilon \to 0, \tag{1.16}$$

where μ_{f_λ} is the absolutely continuous invariant measure of f_λ.

2. Auxiliary lemmas about the transformations f_λ

The following result was proved in Theorem 1.3 of [6].

Lemma 2.1. *Assume f_λ has the form* (1.8), *has no stable periodic orbit and* (1.15) *is true. Then for any $\rho > 0$ there exists $M_\rho > 0$ such that if*

$$|f_\lambda^i x - \tfrac{1}{2}| \geqq \rho \qquad \text{for all } i = 0, 1, \dots, M_\rho - 1 \tag{2.1}$$

then

$$(f_\lambda^{M_\rho})'(x) \geqq \gamma_0 \tag{2.2}$$

for some $\gamma_0 > 1$ independent of x and ρ.

We shall need also the assertion which is complementary to the above lemma. Its proof was communicated to us by M. Misiurewicz.

Lemma 2.2. *For any $\delta > 0$ there exists c_δ bounded away from 0 if $\delta > \tilde{\delta} > 0$ for some $\tilde{\delta} > 0$, such that if $x \in I$ and*

$$\operatorname{dist}(f_\lambda^n x, \mathcal{T}_\lambda) \geqq \delta \tag{2.3}$$

then

$$|(f_\lambda^n)'(x)| \geqq c_\delta \gamma_1^n \tag{2.4}$$

for any $n \geqq 1$, where $\gamma_1 > 1$ is independent of x, δ and n.

Proof. Fix $\rho_0 > 0$ small enough. Let $y_l = f_\lambda^l x$ and $i_1 < \cdots < i_k$ be the numbers when $y_{i_j} \in U_{\rho_0}(\frac{1}{2})$ and $\hat{i}_j = \min\{l > i_j : y_l \notin U_{\rho_0}(f_\lambda^{l-i_j-1}(\frac{1}{2}))\}$. Then

$$|(f_\lambda^{\hat{i}_j - i_j})'(y_{i_j})| \geqq \gamma_2^{\hat{i}_j - i_j} \tag{2.5}$$

for some $\gamma_2 > 1$ independent of x and j, provided ρ_0 was chosen small enough, and where ( )$'$ denotes differentiation. Indeed

$$|f_\lambda'(y_{i_j})| = 8\lambda\, |\tfrac{1}{2} - y_{i_j}| \tag{2.6}$$

and

$$\operatorname{dist}(y_{i_j+1}, f(\tfrac{1}{2})) = 4\lambda\, |\tfrac{1}{2} - y_{i_j}|^2. \tag{2.7}$$

Then it is easy to see that, for some $c_3 > 0$ independent of j,

$$C_3 |\tfrac{1}{2} - y_{i_j}|^{-2} \geqq |(f_\lambda^{\hat{i}_j - i_j - 1})'(y_{i_j+1})| \geqq C_3^{-1} |\tfrac{1}{2} - y_{i_j}|^{-2}. \tag{2.8}$$

On the other hand, by Lemma 2.1,

$$\operatorname{dist}(y_{i_j+l}, f_\lambda^l(\tfrac{1}{2})) \geqq \tilde{C}_3^{-1} \tilde{\gamma}_2^{l-1} \operatorname{dist}(y_{i_j+1}, f_\lambda(\tfrac{1}{2})) \tag{2.9}$$

for all $l = 1, \ldots, \hat{i}_j = i_j$ where $\tilde{C}_3 > 0$ and $\tilde{\gamma}_2 > 1$. Therefore $(\hat{i}_j - i_j)$ is of order $\log |\frac{1}{2} - y_{i_j}|^{-1}$ and so (2.8) implies (2.5).

To consider the trajectory between the times $\hat{i}_j$ and i_{j+1} we shall use the following assertion:

$$\text{if } |f_\lambda^i y - \tfrac{1}{2}| > |y - \tfrac{1}{2}| \text{ for } i = 1, \ldots, m-1 \text{ and } |f_\lambda^m y - \tfrac{1}{2}| \leqq |y - \tfrac{1}{2}| \text{ then } |(f_\lambda^m)'(y)| \geqq \gamma_3^m \tag{2.10}$$

for some $\gamma_3 > 1$ independent of y and m. In particular, if $|f_\lambda y - \frac{1}{2}| \leqq |y - \frac{1}{2}|$ then $|f_\lambda'(y)| \geqq \gamma_3$.

Before proving this statement we shall use it for the proof of our lemma.

Since $y_{i_j+l} \notin U_{\rho_0}(\frac{1}{2})$ for $l = 0, \ldots, i_{j+1} - \hat{i}_j - 1$ and $y_{i_{j+1}} \in U_{\rho_0}(\frac{1}{2})$ one can partition the trajectory y_{i_j+l}, $l = 0, \ldots, i_{j+1} - \hat{i}_j$ into the pieces such that the last point in each piece will be closer to $\frac{1}{2}$ than the first one, and all intermediate points will be farther from $\frac{1}{2}$ than the first point. Hence applying the assertion (2.10) we shall get

$$|f_\lambda^{i_{j+1}-\hat{i}_j})'(y_{\hat{i}_j})| \geqq \gamma_3^{i_{j+1}-\hat{i}_j}. \tag{2.11}$$

The same argument is valid for the trajectory $y_0, y_1, \ldots, y_{i_1}$ and so

$$|(f_\lambda^{i_1})'(x)| \geqq \gamma_3^{i_1}. \tag{2.12}$$

Consider the last piece of the trajectory $y_{i_1}, \ldots, y_n$. If $n > \hat{i}_k$ then we use (2.5) with $j = k$ and Lemma 2.1 for $\hat{i}_k, \ldots, n$ to get

$$|(f_\lambda^{n-\hat{i}_k})'(y_{\hat{i}_k})| \leqq C_4^{-1}\tilde{\gamma}_3^{n-\hat{i}_k} \qquad \text{for some } C_4 > 0 \tag{2.13}$$

since $y_{\hat{i}_k+l} \notin U_{\rho_0}(\frac{1}{2})$, $l = 0, \ldots, n$.

If $i_k < n \leqq \hat{i}_k$ then we shall use the arguments similar to the beginning of the proof taking into account (2.3). Then one can see that for some $\tilde{C}_4 > 0$,

$$|(f_\lambda^{n-i_k})'(y_{i_k})| \geqq \tilde{C}_4^{-1}\delta\,|\tfrac{1}{2} - y_{i_k}|^{-1}. \tag{2.14}$$

On the other hand, by (2.9), there is $C_5 > 0$,

$$\delta_0 \geqq C_5^{-1}\tilde{\gamma}_2^{n-i_k-1}|\tfrac{1}{2} - y_{i_k}|^2 \tag{2.15}$$

since $n \leqq \hat{i}_k$. Now (2.14) and (2.15) yield

$$|(f_\lambda^{n-i_k})'(y_{i_k})| \geqq \tilde{C}_5^{-1}\delta(\tilde{\gamma}_2^{1/2})^{n-i_k-1} \qquad \text{with } \tilde{C}_5 = \delta_0^{1/2}C_5\tilde{C}_4. \tag{2.16}$$

Finally, collecting (2.5), (2.11)–(2.13) and (2.16) we obtain (2.4).

Now let us return to the assertion (2.10). First, if (2.10) is true then there exist two periodic points p_1 and p_2 having period m and so that $p_1 \leqq y \leqq p_2$ or $p_1 \leqq y' \leqq p_2$ where $|y - \frac{1}{2}| = |y' - \frac{1}{2}|$. This is proved in Lemma II.5.6. of [3].

The negative Schwarzian derivative yields that

$$|(f_\lambda^m)'(p_1)| \leqq |(f_\lambda^m)'(y)| \leqq |(f_\lambda^m)'(p_2)| \tag{2.17}$$

or, alternatively, that both inequalities are in the opposite direction.

Now it remains to show that there exists $\gamma_4 > 1$ such that for any periodic point y having a period l

$$|(f_\lambda^l)'(y)| \geqq \gamma_4^l. \tag{2.18}$$

The proof is by induction. If $f_\lambda^i y \notin U_{\rho_0}(\frac{1}{2})$ for all $i = 0, \ldots, l$ then we apply Lemma 2.1 to get (2.18) if l is big enough. If $l \leqq l_0$ then there exists only a bounded number of such periodic trajectories and all of them are sources. So one can choose $\gamma_4 > 1$ to satisfy (2.18).

If $f_\lambda^j y \in U_{\rho_0}(\frac{1}{2})$ and so $f_\lambda^{l+j} y \in U_{\rho_0}(\frac{1}{2})$ then one can partition the whole trajectory into pieces in the same way as at the beginning of the proof of this lemma with x replaced by y. Now we repeat the beginning of the proof using the assertion (2.10) for the pieces of the trajectory $y_{i_j}, \dots, y_{i_{j+1}}$ with x replaced by y. Since $i_{j+1} - \hat{i}_j < l$ then we shall need the inequality (2.18) for periodic orbits with periods less than l that completes the proof by induction.

We shall need a version of the shadowing property.

Lemma 2.3. *Suppose that f_λ of the form* (1.5) *has no stable periodic orbit and satisfies* (1.15). *Let $x_0, \dots, x_n$ be an ε^α-pseudo-orbit of f_λ, i.e.*

$$\text{(2.19)} \qquad \operatorname{dist}(f_\lambda x_i, x_{i+1}) \leqq \varepsilon^\alpha, \qquad i = 0, \dots, n-1$$

where dist *is defined by* (1.10) *and $\varepsilon > 0$ is small enough.*

There exists $C_6 > 0$ depending only on f_λ such that if $0 \leqq \beta \leqq \alpha/2$ and

$$\text{(2.20)} \qquad |x_i - \tfrac{1}{2}| \geqq 2C_6 \varepsilon^\beta, \qquad i = 0, \dots, n$$

then one can find a point $y \in I$ so that

$$\text{(2.21)} \qquad \operatorname{dist}(f_\lambda^i y, x_i) \leqq C_6 \varepsilon^{\alpha-\beta}, \qquad i = 0, \dots, n.$$

Proof. Let

$$\text{(2.22)} \qquad \rho_0 = \operatorname{dist}(\tfrac{1}{2}, \mathcal{T}_\lambda),$$

then one can pick $\rho_3 < \frac{1}{2}\rho_0$ such that

$$f_\lambda(U_{2\rho_3}(\tfrac{1}{2})) \cap U_{2\rho_3}(\tfrac{1}{2}) = \varnothing.$$

Let $i_1 < \cdots < i_k$ be such that $x_{i_j} \in U_{\rho_3}(\frac{1}{2})$, $j = 1, \dots, k$ and

$$x_l \notin U_{\rho_3}(\tfrac{1}{2}) \qquad \text{if } l \neq i_j, \quad j = 1, \dots, k.$$

Put also $i_0 = 0$ and $i_{k+1} = n$.

First, Lemma 2.1 enables us to employ the standard argument yielding the shadowing in the expanding case for pieces $x_{i_j+1}, \dots, x_{i_{j+1}}$ of the pseudo-orbit to conclude that there exists $C_\rho > 0$ independent of the pseudo-orbit $x_0, \dots, x_n$ and some points y_j, $j = 0, \dots, k$ such that

$$\text{(2.23)} \qquad \operatorname{dist}(x_{i_j+l}, f_\lambda^l y_j) \leqq C_{\rho_3} \varepsilon^\alpha$$

for all $l = 1, \dots, i_{j+1} - i_j$ and $j = 0, \dots, k$.

Indeed, if ε is small enough then by (2.19) it follows that if $x_i \notin U_{\rho_3}(\frac{1}{2})$ for $i = l+1, \dots, l + M_{3\rho_3/4} - 1$ then $f_\lambda^q x_l \in U_{3\rho_3/4}(\frac{1}{2})$ for $q = 1, \dots, M_{3\rho_3/4} - 1$. Using this argument and Lemma 2.1 one can see easily that one of preimages in $f_\lambda^{-(i_{j+1}-i_j)} x_{i_{j+1}}$ satisfies (2.23).

Next, we shall prove that there exists a point $y \in f_\lambda^{-i_k} y_k$ satisfying (2.21).

By (2.19) and (2.23) it follows that

$$\text{dist}(f_\lambda x_{i_j}, f_\lambda y_j) \leqq (C_{\rho_3}+1)\varepsilon^\alpha. \tag{2.24}$$

Put $r_j = x_{i_j} - \frac{1}{2}$, then $f_\lambda x_{i_j} = \lambda - 4\lambda r_j^2$ and so by (2.24),

$$4\lambda y_j(1-y_j) = \lambda - 4\lambda r_j^2 + q_\varepsilon^{(j)}\varepsilon^\alpha$$

with

$$|q_\varepsilon^{(j)}| \leqq C_{\rho_3} + 1.$$

Therefore one can take

$$y_j = \tfrac{1}{2} + r_j\left(1 - \frac{q_\varepsilon^{(j)}\varepsilon^\alpha}{4\lambda r_j^2}\right)^{1/2}. \tag{2.25}$$

Since $|r_j| \geqq 2C_6\varepsilon^\beta$ and $2\beta \geqq \alpha$, then by (2.23) if C_6 is chosen big enough,

$$\text{dist}(y_j, f_\lambda^{i_j - i_{j-1}} y_{j-1}) \leqq \tilde{C}_{\rho_3}\frac{\varepsilon^\alpha}{|r_j|} \tag{2.26}$$

for some $\tilde{C}_{\rho_3} > 0$ independent of ε, j and the points $\{x_i\}$ and $\{y_j\}$.

Since $|r_j| \geqq 2C_6\varepsilon^\beta$ then for ε small enough it follows from (2.23) and (2.26) that

$$\text{dist}(y_j, \tfrac{1}{2}) \leqq \tfrac{2}{3}\rho_3 \quad \text{and} \quad \text{dist}(f_\lambda^{i_j - i_{j-1}} y_{j-1}, \tfrac{1}{2}) \leqq \tfrac{2}{3}\rho_3$$

and so

$$\text{dist}(y_j, \mathcal{T}_\lambda) > \tfrac{1}{3}\rho_0 \quad \text{and} \quad \text{dist}(f_\lambda^{i_j - i_{j-1}} y_{j-1}, \mathcal{T}_\lambda) > \tfrac{1}{3}\rho_0. \tag{2.27}$$

Therefore one can employ Lemma 2.2 to obtain that

$$\text{dist}(f_\lambda^{-l} y_j, f_\lambda^{i_j - i_{j-1} - l} y_{j-1}) \leqq C_7\tilde{C}_{\rho_3}\frac{\varepsilon^\alpha}{|r_j|}\gamma_0^{-l} \tag{2.28}$$

for appropriate preimages of y_j where $l \geqq 0$ and $C_7 > 0$ depends only on ρ_0 in (2.22).

It follows from here that

$$\text{dist}(f_\lambda^{-(i_k - i_j)+l} y_k, f_\lambda^l y_j) \leqq \tilde{C}\rho_3\varepsilon^{\alpha-\beta}\sum_{r=j}^{k} C_7^{r-j}\gamma_0^{-(i_r - i_j + l)} \tag{2.29}$$

for corresponding preimages of y_k where $l = 1, \dots, i_{j+1} - i_j$.

It is easy to see that

$$i_{j+1} - i_j \geqq C_8^{-1}\log\rho_3^{-1} \tag{2.30}$$

where $C_8 > 0$ is independent of ρ_3, i_j and the choice of points $\{x_i\}$ and $\{y_j\}$. Therefore if ρ_3 was taken initially small enough then $C_7\gamma_0^{-(i_{j+1}-i_j)} < 1$ and the sum in the right hand side of (2.29) is bounded. This together with (2.23) and (2.29) yield (2.21) for some $y \in f^{-i_k} y_k$ and proves Lemma 2.3.

Next we shall need the following

Lemma 2.4. *Let $J \subset I$ be an interval such that f_λ^n maps J homeomorphically* *onto its image* $Q = f_\lambda^n J$ *and*

$$\inf_{y\in Q} \operatorname{dist}(y, \mathcal{T}_\lambda) = \delta > 0, \tag{2.31}$$

then

$$\frac{\sup_{x\in J} |(f_\lambda^n)'(x)|}{\inf_{x\in J} |(f_\lambda^n)'(x)|} \leqq K_\delta \tag{2.32}$$

where $K_\delta > 0$ *is independent of* n *and the interval* J.

Proof. Let $J = (a, b)$, i.e. a and b are the endpoints of J and denote $a_l = f_\lambda^l a$ and $b_l = f_\lambda^l b$. Then by Lemma 2.2

$$\operatorname{dist}(a_l, b_l) \leqq c_\delta^{-1} \gamma_0^{-(n-l)} \operatorname{dist}(a_n, b_n). \tag{2.33}$$

Let ρ_0 be defined by (2.22) and take $\rho_4 < \frac{1}{4}\rho_0$ small enough. Without loss of generality we can assume that $\delta < \frac{1}{4}\rho_0$ and

$$\operatorname{mes} Q \leqq \rho_4 \leqq \delta c_\delta \tag{2.34}$$

where mes denotes Lebesgue measure.

Let $i_1 < \cdots < i_k$ be such that either $a_{i_j} \in U_{\rho_4}(\frac{1}{2})$ or $b_{i_j} \in U_{\rho_4}(\frac{1}{2})$, $j = 1, \ldots, k$ but $a_l \notin U_{\rho_4}(\frac{1}{2})$ and $b_l \notin U_{\rho_4}(\frac{1}{2})$ if $l \neq i_j$ for some $j = 1, \ldots, k$. Let

$$\nu_\delta^{(j)} = \min\{\nu > i_j + 1 : \operatorname{dist}(f_\lambda^{\nu - i_j} a_{i_j}, f_\lambda^{\nu - i_j}(\tfrac{1}{2})) \geqq \delta/4 \text{ and } \operatorname{dist}(f_\lambda^{\nu - i_j} b_{i_j}, f_\lambda^{\nu - i_j}(\tfrac{1}{2})) \geqq \delta/4\}, \tag{2.35}$$

then

$$\text{either} \quad \operatorname{dist}(a_l, f_\lambda^l(\tfrac{1}{2})) \leqq \delta \quad \text{or} \quad \operatorname{dist}(b_l, f_\lambda^l(\tfrac{1}{2})) \leqq \delta \tag{2.36}$$

for $l = i_j + 1, \ldots, \nu_\delta^{(j)}$. By (2.33) and (2.34), $\operatorname{mes}(a_l, b_l) \leqq \delta$ and so

$$a_l \notin U_{\rho_0/2}(\tfrac{1}{2}) \quad \text{and} \quad b_l \notin U_{\rho_0/2}(\tfrac{1}{2}) \qquad \text{for } l = i_j + 1, \ldots, \nu_\delta^{(j)}. \tag{2.37}$$

Let σ_0 be the Lipschitz constant of $\log|f_\lambda'|$ on $I \setminus U_{\rho_0/2}(\frac{1}{2})$. Denote by L_j the minimal interval containing points a_{i_j+1}, b_{i_j+1} and $f_\lambda(\frac{1}{2})$. Then

$$\begin{aligned} \log \frac{\sup_{z\in L_j} |(f_\lambda^{\nu_\delta^{(j)} - i_j - 1})'(z)|}{\inf_{z\in L_j} |(f_\lambda^{\nu_\delta^{(j)} - i_j - 1})'(z)|} &= \sup_{z_1, z_2 \in L_j} \sum_{0 \leqq l \leqq \nu_\delta^{(j)} - i_j - 1} (\log|f_\lambda'(f^l(z_1))| - \log|f_\lambda'(f^l(z_2))|) \\ &\leqq \sigma_0 \sum_{0 \leqq l \leqq \nu_\delta^{(j)} - i_j - 1} \operatorname{mes}(f_\lambda^l L_j). \end{aligned} \tag{2.38}$$

Since by (2.33) and (2.34)

$$\operatorname{mes}(a_{\nu_\delta^{(j)}}, b_{\nu_\delta^{(j)}}) \leqq \delta \tag{2.39}$$

then by (2.36),

$$\operatorname{mes}(f_\lambda^{\nu_\delta^{(j)} - i_j - 1} L_j) \leqq 2\delta \tag{2.40}$$

and

$$\operatorname{dist}(\tfrac{1}{2}, f_\lambda^l L_j) > \tfrac{1}{2}\rho_0 \qquad \text{for all } l = 0, \dots, \nu_\delta^{(j)} - i_j - 1. \tag{2.41}$$

Therefore by Lemma 2.1,

$$\operatorname{mes}(f_\lambda^l L_j) \leqq C_9 \delta \gamma_3^{(-\nu_\delta^{(j)} - i_j - l)} \qquad \text{for } l = 0, \dots, \nu_\delta^{(j)} - i_j - 1 \tag{2.42}$$

where $C_9 > 0$ and $\gamma_3 > 1$ depend only on ρ_0.

Hence the right hand side of (2.38) is bounded by $\sigma_0 C_9 \delta (1 - \gamma_3^{-1})^{-1}$. Thus by (2.35), (2.36) and (2.20),

$$\operatorname{mes}(a_{i_j+1}, b_{i_j+1}) \leqq C_{10} \delta^{-1} \operatorname{dist}(f(\tfrac{1}{2}), (a_{i_j+1}, b_{i_j+1})) \operatorname{mes}(a_{\nu_\delta^{(j)}}, b_{\nu_\delta^{(j)}}) \tag{2.43}$$

for some $C_{10} > 0$ depending only on ρ_0.

Solving the equation $4\lambda x(1-x) = z$ with respect to x for z close to λ and taking into account that $\frac{1}{2} \notin (a_{i_j}, b_{i_j})$ because of (2.31) one obtains from (2.43),

$$\operatorname{mes}(a_{i_j}, b_{i_j}) \leqq C_{11} \delta^{-1} \operatorname{mes}(a_{\nu_\delta^{(j)}}, b_{\nu_\delta^{(j)}}) \operatorname{dist}(\tfrac{1}{2}, (a_{i_j}, b_{i_j})) \tag{2.44}$$

for some $C_{11} > 0$.

Again, since $\frac{1}{2} \notin (a_{i_j}, b_{i_j})$ it follows from here that

$$\frac{\sup_{z \in (a_{i_j}, b_{i_j})} |f_\lambda'(z)|}{\inf_{z \in (a_{i_j}, b_{i_j})} |f_\lambda'(z)|} \leqq (1 + C_{12} \delta^{-1} \operatorname{mes}(a_{\nu_\delta^{(j)}}, b_{\nu_\delta^{(j)}})) \tag{2.45}$$

for some $C_{12} > 0$.

Let σ_1 be the Lipschitz constant of $\log |f_\lambda'|$ on $I \setminus U_{\rho_4}(\frac{1}{2})$, then in the same way as in (2.38) one can see that

$$\log \frac{\sup_{z \in (a_{i_j+1}, b_{i_j+1})} |(f_\lambda^{i_{j+1} - i_j - 1})'(z)|}{\inf_{z \in (a_{i_j+1}, b_{i_j+1})} |(f_\lambda^{i_{j+1} - i_j - 1})'(z)|} \leqq \sigma_1 \sum_{i_{j+1} \geqq l \geqq i_j + 1} \operatorname{mes}(a_l, b_l). \tag{2.46}$$

Finally, from (2.33), (2.45) and (2.46) we get (2.32) with

$$K_\delta = \exp\{(c_8 + C_{12}\delta^{-1})(1 - \gamma_0^{-1})^{-1} \operatorname{mes} Q\} \tag{2.47}$$

that completes the proof of Lemma 2.4.

The next result will be important in the proof of the Theorem.

Lemma 2.5. *Let $x \in I$ be a point, $\rho > 0$ and $Q \subset I$ be an interval such that $f_\lambda^{-n}Q$ is defined and* (2.31) *holds. In each connected component J_i, $i = 1, \dots, l$ of the intersection $f_\lambda^{-n}Q \cap U_\rho(x)$ take an arbitrary point y_i, $i = 1, \dots, l$. Then there exists Z_δ independent of n, Q, ρ and the points $\{y_i\}$ but depending on δ in* (2.31) *such that*

$$Z_\delta\left(1 + \rho^{-1} \max_{1 \leqq i \leqq l} |(f_\lambda^n)'(y_i)|^{-1}\right) \geqq \rho^{-1} \sum_{1 \leqq i \leqq l} |(f_\lambda^n)'(y_i)|^{-1}. \tag{2.48}$$

If $n \geqq |\log \rho|^{3/2}$ and ρ is small enough then

$$2Z_\delta \geqq \rho^{-1} \sum_{1 \leqq i \leqq l} |(f_\lambda^n)'(y_i)|^{-1}. \tag{2.49}$$

Proof. By (2.31) one can find an interval $R \supset Q$ with

$$\inf_{z \in R} \operatorname{dist}(z, \mathcal{T}_\lambda) = \delta/2 \quad \text{and} \quad \operatorname{mes} R \geqq \delta/2. \tag{2.50}$$

It follows from Lemma 2.4 that if Γ_1 is a connected component of $f_\lambda^{-n}R$, Γ_2 is a connected component of $f_\lambda^{-n}Q$ and $\Gamma_1 \supset \Gamma_2$ then

$$\frac{\operatorname{mes} \Gamma_2}{\operatorname{mes} \Gamma_1} \leqq K_\delta \frac{\operatorname{mes} Q}{\operatorname{mes} R} \leqq 2\delta^{-1} K_\delta \operatorname{mes} Q. \tag{2.51}$$

Let

$$d = \rho^{-1} \max_{1 \leqq i \leqq l} |(f_\lambda^n)'(y_i)|^{-1},$$

then each connected component Γ of $f_\lambda^{-n}R$ satisfies $\operatorname{mes} \Gamma \leqq \rho d$ and so by (2.50) and (2.51),

$$\frac{\operatorname{mes}(f_\lambda^{-n}Q \cap U_{\rho(1+d)}(x))}{\operatorname{mes}(f_\lambda^{-n}R \cap U_{\rho(1+2d)}(x))} \leqq K_\delta \operatorname{mes} Q. \tag{2.52}$$

Again using Lemma 2.4 one can see that

$$\operatorname{mes}(f_\lambda^{-n}Q \cap U_{\rho(1+d)}(x)) \geqq K_\delta^{-1} \operatorname{mes} Q \sum_{1 \leqq i \leqq l} |(f_\lambda^n)'(y_i)|^{-1} \tag{2.53}$$

where $\{y_i\}$ are the same as in (2.48).

Since $\operatorname{mes}(f_\lambda^{-n}R \cap U_{\rho(1+2d)}(x)) \leqq 2\rho(1+2d)$ then, by (2.52) and (2.53),

$$2C_{13}K_\delta^2(1+2d) \geqq \rho^{-1} \sum_{1 \leqq i \leqq l} |(f_\lambda^n)'(y_i)|^{-1}$$

proving (2.48) with $Z_\delta = 8\delta^{-1}K_\delta^2$.

If $n \geqq |\ln \rho|^{3/2}$ and ρ is small enough then, by Lemma 2.2, it follows that $d \leqq 1$ proving (2.49) and Lemma 2.5.

Employing Lemma 2.1 instead of Lemma 2.2 one can prove the following results similar to Lemmas 2.4 and 2.5.

Lemma 2.6. *Let $J \subset I$ be an interval such that f_λ^n maps J homeomorphically onto its image $Q = f_\lambda^n J$ and*

$$\min_{0 \leq i \leq n-1} \operatorname{dist}(\tfrac{1}{2}, f_\lambda^i J) \geq \rho, \tag{2.54}$$

then

$$\frac{\sup_{x \in J} |(f_\lambda^n)'(x)|}{\inf_{x \in J} |(f_\lambda^n)'(x)|} \leq \tilde{K}_\rho \tag{2.55}$$

where $\tilde{K}_\rho > 0$ is independent of n and J.

Proof. From Lemma 2.1 it follows that

$$\operatorname{mes} f^i J \leq \tilde{C}_\rho (\gamma_1^{1/M_\rho})^{-(n-i)} \tag{2.56}$$

where $\tilde{C}_\rho > 0$ depends only on ρ.

If σ_ρ is the Lipschitz constant of $\log |f_\lambda'|$ on $I \setminus U_\rho(\frac{1}{2})$ then, in the same way as in (2.38), one can see that

$$\frac{\sup_{x \in J} |(f_\lambda^n)'(x)|}{\inf_{x \in J} |(f_\lambda^n)'(x)|} \leq \sigma_\rho \sum_{i=0}^{n} \operatorname{mes} f^i J. \tag{2.57}$$

This together with (2.56) give (2.55) proving Lemma 2.6.

Lemma 2.7. *Let $\delta, \rho > 0$, $n \geq |\ln \rho|^{3/2}$, $x \in I \setminus U_\delta(\frac{1}{2})$ and $Q \subset I \setminus U_\delta(\frac{1}{2})$ be an interval. In each connected component J_j, $j = 1, \ldots, l$ of the intersection $f_\lambda^{-n} Q \cap U_\rho(x)$ satisfying the condition*

$$\min_{0 \leq i \leq n-1} \operatorname{dist}(\tfrac{1}{2}, f_\lambda^i J_j) \geq \delta \tag{2.58}$$

take arbitrary points $y_j \in J_j$, $j = 1, \ldots, l$. Then

$$\tilde{Z}_\delta \geq \rho^{-1} \sum_{1 \leq j \leq l} |(f_\lambda^n)'(y_j)|^{-1} \tag{2.59}$$

where $\tilde{Z}_\delta$ depends only on δ, but independent of n, Q, the points $\{y_j\}$ and $\rho > 0$ provided ρ is small enough.

The proof is the same as in Lemma 2.5 by using Lemma 2.1 and Lemma 2.6 in place of Lemma 2.2 and Lemma 2.4, respectively.

3. Linearized Markov chains

The following result will be proved in the Appendix.

Proposition 3.1. *For arbitrary points $x_1,\dots,x_n \in I$ let $\theta_1,\dots,\theta_n$ be independent random variables with distribution functions $P\{\theta_i \leqq \eta\} = \int_{-\infty}^{\eta} r_{x_i}(\xi)d\xi$. Then there exist C_{13}, $\alpha_3 > 0$ independent of $x_1,\dots,x_n$ and n such that for any non-zero numbers $a_1,\dots,a_n$ the distribution function of the random variable*

$$\left(\sum_{1\leqq i\leqq n} a_i^2\right)^{-1/2} \sum_{1\leqq i\leqq n} a_i(\theta_i - E\theta_i)$$

has the derivative, i.e. the probability density function, satisfying

$$r_{x_1,\dots,x_n}^{a_1,\dots,a_n}(\eta) \leqq C_{13}e^{-\alpha_3|\eta|}$$

where $E\theta_i = \int_{-\infty}^{\infty} \xi r_{x_i}(\xi)d\xi$ is the expectation of θ_i.

For $x \in (0,1)$, $\xi \in R^1 = (-\infty,\infty)$ and a Borel set $\Psi \subset R^1$ define

$$R_x^\varepsilon(\xi,\Psi) = \varepsilon^{-1}\int_{\Psi} r_{f_\lambda x}\left(\frac{\eta - f'_\lambda(x)\xi}{\varepsilon}\right)d\eta \tag{3.1}$$

where the functions $r_x(\xi)$ were introduced at the beginning of §1.

Consider the Markov chain $\zeta_x^\varepsilon(n)$ with the initial condition $\zeta_x^\varepsilon(0) = \xi$ and the transition probability $R_x^\varepsilon(\eta,\Psi)$, i.e.

$$P\{\zeta_x^\varepsilon(n+1) \in \Psi \mid \zeta_x^\varepsilon(n)\} = R_{f_\lambda^n x}^\varepsilon(\zeta_x^\varepsilon(n),\Psi) \tag{3.2}$$

where $P\{\cdot \mid \cdot\}$ denotes the corresponding conditional probability.

Lemma 3.1. *Let $\{\theta_x(k) \in R^1,\ k = 1,\dots\}$ be mutually independent random variables with the distributions*

$$P\{\theta_x(k) \in \Psi\} = \int_{\Psi} r_{f_\lambda^k x}(\eta)d\eta. \tag{3.3}$$

Then one can write

$$\zeta_x^\varepsilon(n) = \varepsilon\sum_{k=1}^{n} (f_\lambda^{n-k})'(f_\lambda^k x)\theta_x(k) + (f_\lambda^n)'(x)\xi \tag{3.4}$$

in the sense that the left and right hand sides of (3.4) *have the same multidimensional distributions and so probabilities of all events for both sides of* (3.4) *are the same.*

Proof. Denote by $\tilde\zeta_x^\varepsilon(n)$ the right hand side of (4.4), then

$$\begin{aligned} P\{\tilde\zeta_x(n) \in \Psi \mid \tilde\zeta_x^\varepsilon(n-1)\} &= P\{\varepsilon\theta(n) + f'_\lambda(f_\lambda^{n-1}x)\tilde\zeta_x^\varepsilon(n-1) \in \Psi \mid \tilde\zeta_x^\varepsilon(n-1)\} \\ &= R_{f_\lambda^{n-1}x}(\tilde\zeta_x^\varepsilon(n-1),\Psi), \end{aligned} \tag{3.5}$$

since $\theta_x^\varepsilon(n)$ and $\tilde\zeta_x^\varepsilon(n-1)$ are independent. But $\zeta_x^\varepsilon(n)$ and $\tilde\zeta_x^\varepsilon(n)$ are both Markov chains $\zeta_x^\varepsilon(0) = \tilde\zeta_x^\varepsilon(0)$ and by (3.2) and (3.5) they have the same transition probabilities. Therefore ζ_x^ε and $\tilde{\zeta}_x^\varepsilon$ have the same multidimensional distributions and the proof is completed.

For $\xi \in R^1$ and Borel set $\Psi \subset R^1$ denote by $R_x^\varepsilon(n, \xi, \Psi)$ the probability of the event $\{\zeta_x^\varepsilon(n) \in \Psi\}$ provided $\zeta_x^\varepsilon(0) = \xi$. By the Chapman–Kolmogorov formula

$$R_x^\varepsilon(n, \xi, \Psi) = \int_{R^1} \cdots \int_{R^1} r_x^\varepsilon(\xi, \eta_1) r_{f_\lambda x}^\varepsilon(\eta_1, \eta_2) \cdots r_{f_\lambda^n x}^\varepsilon(\eta_{n-1}, \eta_n) d\eta_1 \cdots d\eta_n \tag{3.6}$$

where

$$r_y^\varepsilon(\eta, \zeta) = \varepsilon^{-1} r_{f_\lambda y}\left(\frac{\zeta - f_\lambda^1(y)\eta}{\varepsilon}\right). \tag{3.7}$$

Lemma 3.2. *There exists $C_{14} > 0$ such that for any $x \in (0,1)$, $\xi \in R^1$ and a Borel set $\Psi \subset \{\eta : |\eta| \leqq 1\}$ one has*

$$R_x^\varepsilon(n, \xi, \Psi) \leqq C_{14} \varepsilon^{-1} B_n^{-1/2}(x) \operatorname{mes} \Psi \, |(f_\lambda^n)'(x)|^{-1} \tag{3.8}$$

provided $f_\lambda^k x \neq \frac{1}{2}$ for all $k = 0, \ldots, n-1$, where $B_n(x)$ is defined in (3.13) *below.*

If, in addition, $n \geqq (\ln \varepsilon)^2$ and

$$\operatorname{dist}(f_\lambda^n x, \mathcal{T}_\lambda) \geqq \delta \tag{3.9}$$

then

$$R_x^\varepsilon(n, \xi, \Psi) \leqq C_{14} \varepsilon^{-1} B_n^{-1/2}(x) \operatorname{mes} \Psi \, |(f_\lambda^n)'(x)|^{-1} \exp\{-\alpha_3 B_n^{-1/2}(x) |\varepsilon^{-1}\xi + \beta_x(n)|\} \tag{3.10}$$

provided $\varepsilon > 0$ is small enough with respect to δ, where $\beta_x^{(n)}$ is defined by (3.11) *below.*

Proof. Define the random variables

$$\varphi_x(n) = \sum_{k=1}^n ((f_\lambda^k)'(x))^{-1} (\theta_x(k) - b(f_\lambda^k(x))),$$

where $b(y) = \int_{-\infty}^{\infty} \xi r_y(\xi) d\xi$ and

$$\beta_x(n) = \sum_{k=1}^n ((f_\lambda^k)'(x))^{-1} b(f_\lambda^k x). \tag{3.11}$$

Then by (3.4),

$$\begin{aligned} R_x^\varepsilon(n, \xi, \Psi) &= P\{(f_\lambda^n)'(x)(\varepsilon\varphi_x(n) + \varepsilon\beta_x(n) + \xi) \in \Psi\} \\ &= P\{B_n^{-1/2}(x)(\varphi_x(n) + \beta_x(n) + \varepsilon^{-1}\xi) \in \varepsilon^{-1} B_n^{-1/2}(x)((f_\lambda^n)'(x))^{-1}\Psi\} \end{aligned} \tag{3.12}$$

where

$$B_n(x) = \sum_{k=1}^n ((f_\lambda^k)'(x))^{-2}. \tag{3.13}$$

By Proposition 3.1 the distribution of the random variable $B_n^{-1/2}(x)\varphi_x(n)$ has a density with an exponentially decreasing estimate and so by (3.12) one obtains

$$R_x^\varepsilon(n,\xi,\Psi) \leqq C_{13}\varepsilon^{-1}B_n^{-1/2}(x)\,|(f_\lambda^n)'(x)|^{-1}\operatorname{mes}\Psi\exp\{-\alpha_3|\eta_n^\varepsilon(x)|\} \tag{3.14}$$

where

$$\eta_n^\varepsilon(x) = B_n^{-1/2}(x)\inf_{\eta\in\varepsilon^{-1}((f_\lambda^n)'(x))^{-1}\Psi}|\varepsilon^{-1}\xi+\beta_x(n)-\eta| \tag{3.15}$$

and (3.8) follows.

Next assume that (3.9) is true and $n \geqq (\ln\varepsilon)^2$. Then for ε small enough one can see from Lemma 2.2 that

$$\sup_{\eta\in\varepsilon^{-1}((f_\lambda^n)')^{-1}\Psi}|\eta| \leqq 1. \tag{3.16}$$

Since

$$B_n(x) \geqq (f_\lambda'(x))^{-2} \geqq (4\lambda)^{-2} \tag{3.17}$$

one obtains (3.10) from (3.14)-(3.17) proving Lemma 3.2.

Now we shall prove a similar result under the conditions of Lemma 2.1 instead of Lemma 2.2.

Lemma 3.3. *There exist C_{15}, $\alpha_4 > 0$ such that for any $x\in(0,1)$, $\xi\in R^1$ and a Borel set $\Psi\subset\{\eta: |\eta|\leqq 1\}$ one has*

$$R_x^\varepsilon(n,\xi,\Psi) \leqq C_{22}\varepsilon^{-1}\operatorname{mes}\Psi\,|(f_\lambda^n)'(x)|^{-1}\exp\{-\alpha_4\varepsilon^{-1}|\xi|\} \tag{3.18}$$

provided $n\geqq(\ln\varepsilon)^2$, $\varepsilon>0$ is small enough and

$$\min_{0\leqq k\leqq n-1}\operatorname{dist}(f_\lambda^k x, \tfrac12) \geqq \tfrac12\rho_0 \tag{3.19}$$

where $\rho_0 = \frac14\operatorname{dist}(\frac12,\mathcal{T}_\lambda)$. If $\Psi\subset\{\eta: |\eta|\leqq\varepsilon\}$ then (3.18) is true for any $n\geqq 1$.

Proof. By Lemma 2.1

$$|(f_\lambda^k)'(x)|^{-1} \leqq C_{16}(\gamma_1^{1/M_\delta})^{-k} \qquad \text{for all } k=0,\dots,n-1 \tag{3.20}$$

with some $C_{16}>0$. Thus (3.16) and (3.27) follow and the proof is the same as in Lemma 3.2 since in our circumstances $C_{17}^{-1}\leqq B_n(x)\leqq C_{17}$ for $C_{17}>0$ independent of n and x.

Since we have identified the points 0 and 1, then the right and left hand side derivatives of f_λ at 0 need not coincide. For this reason we shall construct another Markov chain which will describe the behaviour of the initial process x_n^ε when it stays near zero.

For $\xi\in R^1$ and a Borel set $\Psi\subset R^1$ define

$$R_0^\varepsilon(\xi,\Psi)=\varepsilon^{-1}\int_\Psi r_0\left(\frac{\eta-4\lambda|\xi|}{\varepsilon}\right)d\eta \tag{3.21}$$

where

$$4\lambda=f_\lambda'(x)>2,\quad \text{i.e. } \lambda>\tfrac{1}{2} \tag{3.22}$$

since for $\lambda\leqq\frac{1}{2}$ the transformation f_λ maps $[0,\frac{1}{2}]\to[0,\lambda]$ homeomorphically and so it has a stable periodic orbit that contradicts our conditions.

It is easy to see in the same way as in Lemma 3.1 that $R_0(\xi,\Psi)$ is the transition probability function of the Markov chain ζ_n^ε constructed inductively in the following way: $\zeta_0^\varepsilon=\xi$,

$$\zeta_n^\varepsilon=4\lambda\,|\zeta_{n-1}^\varepsilon|+\varepsilon\theta_n, \tag{3.23}$$

where $\{\theta_k\in R^1,\ k=1,\dots\}$ are independent random variables with the distribution

$$P\{\theta_k\in\Psi\}=\int_\Psi r_0(\eta)d\eta. \tag{3.24}$$

Let $R_0^\varepsilon(n,\xi,\Psi)$ be the probability of the event $\{\zeta_n\in\Psi\}$ provided $\zeta_0^\varepsilon=\xi$. By the Chapman–Kolmogorov formula

$$R_0^\varepsilon(n,\xi,\Psi)=\int_{R^1}\cdots\int_{R^1} r_0^\varepsilon(\xi,\eta_1)r_0^\varepsilon(\eta_1,\eta_2)\cdots r_0^\varepsilon(\eta_{n-1},\eta_n)d\eta_1\cdots d\eta_n \tag{3.25}$$

where

$$r_0^\varepsilon(\eta,\zeta)=\varepsilon^{-1}r_0\left(\frac{\zeta-4\lambda|\eta|}{\varepsilon}\right). \tag{3.26}$$

Lemma 3.4. *For any n, $\varepsilon>0$ and a Borel set Ψ,*

$$R_0^\varepsilon(n,\xi,\Psi)\leqq C_0\varepsilon^{-1}(2\lambda)^{-(n-1)}\operatorname{mes}\Psi \tag{3.27}$$

where $C_0>0$ is the same as in (1.1).

Proof. Put $\varphi_n=\varepsilon^{-1}(4\lambda)^{-n}\zeta_n^\varepsilon$, then

$$\varphi_n=|\varphi_{n-1}|+(4\lambda)^{-n}\theta_n\quad\text{and}\quad\varphi_0=\varepsilon^{-1}\xi. \tag{3.28}$$

Notice that if η is a random variable with some distribution having a density $p_\eta(x)$ satisfying $p_\eta(x)\leqq\tilde C$ for all $x\in R^1$ and some constant $\tilde C>0$ then the random variable $|\eta|$ has the distribution with a density $p_{|\eta|}(x)$ satisfying $p_{|\eta|}(x)\leqq 2\tilde C$. Indeed,

$$P\{|\eta|\in\Gamma\}\leqq P\{\eta\in\Gamma\}+P\{\eta\in-\Gamma\}$$

and since $\operatorname{mes}\Gamma = \operatorname{mes}(-\Gamma)$ then

$$p_{|\eta|}(x) \leqq p_\eta(x) + p_\eta(-x) \leqq 2\tilde{C}. \tag{3.29}$$

Consider $\varphi_1 = \varepsilon^{-1}|\xi| + (4\lambda)^{-1}\theta_1$. By (1.1) and (3.24) the distribution of φ_1 has the density

$$p_{\varphi_1}(x) = 4\lambda r_0(4\lambda(x - \varepsilon^{-1}|\xi|)) \leqq 4\lambda C_0. \tag{3.30}$$

Since φ_{k-1} and θ_k are independent, then by the argument above and (4.41), if the density $p_{\varphi_{k-1}}(x) \leqq C^{(k-1)}$ for all x, then

$$p_{\varphi_k}(x) = \int_{R^1} P\{(4\lambda)^{-k}\theta_k \in dy\} p_{|\varphi_{k-1}|}(x - y) \leqq 2C^{(k-1)} \tag{3.31}$$

for all $k = 2, \dots, n$. From (3.30) and (3.31) it follows that

$$p_{\varphi_n}(x) \leqq \lambda 2^{n+1} C_0. \tag{3.32}$$

Therefore

$$\begin{aligned} R_0^\varepsilon(n, \xi, \Psi) &= P\{\zeta_n^\varepsilon \in \Psi\} = P\{\varphi_n^\varepsilon \in \varepsilon^{-1}(4\lambda)^{-n}\Psi\} \\ &= \int_{\varepsilon^{-1}(4\lambda)^{-n}\Psi} p_{\varphi_n^\varepsilon}(x)dx \leqq C_0\varepsilon^{-1}(2\lambda)^{-(n-1)} \operatorname{mes}\Psi \end{aligned} \tag{3.33}$$

proving (3.27).

We shall need also

Lemma 3.5. *For any n, m, $\varepsilon > 0$ and a Borel set Ψ,*

$$\begin{aligned} \tilde{R}_x^\varepsilon(n, m, \Psi) &\equiv \int_{-\infty}^{\infty} R_0(n, \xi, d\eta) R_x(m, \eta, \Psi) \\ &\leqq C_0\varepsilon^{-1}|(f_\lambda^m)'(x)|^{-1}(2\lambda)^{-(n-1)} \operatorname{mes}\Psi. \end{aligned} \tag{3.34}$$

Proof. It is easy to see that $\tilde{R}_x^\varepsilon(n, m, \Psi)$ is the probability $P\{\kappa_{n,m}^\varepsilon \in \Psi\}$ for the process

$$\kappa_{n,m}^\varepsilon = \varepsilon \sum_{k=1}^m (f_\lambda^{m-k})'(f_\lambda^k x)\theta_x(k) + (f_\lambda^m)'(x)\zeta_n^\varepsilon, \tag{3.35}$$

where $\{\theta_x(k)\}$ are the same as in Lemma 3.1 and these random variables are chosen to be independent of ζ_n^ε defined by (3.23).

Let

$$\eta_x(m) = \sum_{k=1}^m ((f_\lambda^k)'(x))^{-1}\theta_x(k).$$

Since $\eta_x(m)$ and φ_n in (3.41) are independent then, by (3.45),

$$\tilde{R}^{\varepsilon}_x(n,m,\Psi) = P\{\kappa^{\varepsilon}_{n,m}\in\Psi\}$$

$$(3.36)\qquad = P\{(4\lambda)^{-n}\eta_x(m)+\varphi_n\in\varepsilon^{-1}((f^m_\lambda)'(x))^{-1}(4\lambda)^{-n}\Psi\}$$

$$\leqq C_0\varepsilon^{-1}|(f^m_\lambda)'(x)|^{-1}(2\lambda)^{-(n-1)}\,\mathrm{mes}\,\Psi$$

proving Lemma 3.5.

4. Proof of the Theorem

If $\Gamma\subset I$ is a Borel set define

$$J^{\varepsilon}_1(\rho,n,x,\Gamma) = P_x\left\{\min_{0\leqq k\leqq n-1}\mathrm{dist}(x^{\varepsilon}_k,\tfrac{1}{2})>\rho \text{ and } x^{\varepsilon}_n\in\Gamma\right\}$$

$$(4.1)\qquad = \int_{I\setminus U_\rho(\frac12)}\cdots\int_{I\setminus U_\rho(\frac12)}\int_{\Gamma} q^{\varepsilon}_{f_\lambda x}(y_1)q^{\varepsilon}_{f_\lambda y_1}(y_2)\cdots q^{\varepsilon}_{f_\lambda y_{n-1}}(y_n)\,dy_1\cdots dy_n$$

where we have used the Chapman–Kolmogorov formula, $q^{\varepsilon}_y(z)$ is defined in §1 and $P_x\{\cdot\}$ denotes the probability of the event in brackets provided $x^{\varepsilon}_0 = x$.

The main step in the proof of the Theorem is the following.

Lemma 4.1. *There exists $\gamma_0>0$ such that for any $x\in[\varepsilon^{\alpha_2+\gamma_0},1-\varepsilon^{\alpha_2+\gamma_0}]$ and an interval $Q\subset I$ satisfying* (2.31) *with some $\delta>0$ one has*

$$(4.2)\qquad J^{\varepsilon}_1(\varepsilon^{\gamma},n,x,Q)\leqq D_\delta\,\mathrm{mes}\,Q$$

provided $(\ln\varepsilon)^4\geqq n\geqq(\ln\varepsilon)^2$, $\gamma\leqq\gamma_0$ *and ε is small enough, where $D_\delta>0$ depends only on δ and α_2.*

Proof. For any Borel set $\Gamma\subset I$ and any numbers $\rho,\eta>0$ define

$$J^{\varepsilon}_2(\rho,\eta,n,x,\Gamma)$$

$$(4.3)\qquad = P_x\{x^{\varepsilon}_{k+1}\in U_\eta(f_\lambda x^{\varepsilon}_k),\ \mathrm{dist}(x^{\varepsilon}_k,\tfrac12)>\rho \text{ for all } k=0,\dots,n-1 \text{ and } x^{\varepsilon}_n\in\Gamma\}.$$

Then by the Chapman–Kolmogorov formula

$$J^{\varepsilon}_2(\rho,\eta,n,x,\Gamma) = \int_{U_\eta(f_\lambda x)\cap(I\setminus U_\rho(\frac12))}\int_{U_\eta(f_\lambda y_1)\cap(I\setminus U_\rho(\frac12))}\int_{U_\eta(f_\lambda y_{n-2})\cap(I\setminus U_\rho(\frac12))}\int_{U_\eta(f_\lambda y_{n-1})\cap\Gamma}$$

$$(4.4)\qquad \times\, q^{\varepsilon}_{f_\lambda x}(y_1)q^{\varepsilon}_{f_\lambda y_1}(y_2)\cdots q^{\varepsilon}_{f_\lambda y_{n-1}}(y_n)\,dy_1\cdots dy_n.$$

By (1.1), (1.8), (1.9) and (1.11) it is clear that

$$|J^{\varepsilon}_1(\rho,n,x,\Gamma)-J_2(\rho,\eta,n,x,\Gamma)|$$

$$(4.5)\qquad \leqq(1+\varepsilon^{\alpha_2})^2(C_0+1)^2n\varepsilon^{-2}\,\mathrm{mes}\,\Gamma\exp\left(-\frac{\eta}{\varepsilon}\min(\alpha_1,\alpha_2)\right).$$

Let

$$\varepsilon \leqq \operatorname{mes} Q \tag{4.6}$$

where Q is an interval. Then one can find points $v_1, \ldots, v_{k_\varepsilon}$ such that

$$Q \subset \bigcup_{1 \leqq i \leqq k_\varepsilon} U_\varepsilon(v_i) \quad \text{and} \quad \operatorname{mes} Q \geqq \frac{1}{2} \sum_{1 \leqq i \leqq k_\varepsilon} \operatorname{mes} U_\varepsilon(v_i) = \varepsilon k_\varepsilon. \tag{4.7}$$

If $\varepsilon > \operatorname{mes} Q$ then instead of $U_\varepsilon(v_i)$ we shall take Q itself. Clearly

$$J_2^\varepsilon(\rho, \eta, n, x, Q) \leqq \sum_{1 \leqq i \leqq k_\varepsilon} J_2^\varepsilon(\rho, \eta, n, x, U_\varepsilon(v_i)). \tag{4.8}$$

Take

$$\eta = \varepsilon^{1-\beta} \quad \text{and} \quad \rho = 2C_6\varepsilon^\beta \tag{4.9}$$

where $C_6 > 0$ is the same as in (2.20) and $\beta > 0$ will be chosen small enough.

According to (4.4) the integration in $J_2^\varepsilon(2C_6\varepsilon^\beta, \varepsilon^{1-\beta}, n, x, U_\varepsilon(v_i))$ is over $\varepsilon^{1-\beta}$-pseudo-orbits $\omega = (x, y_1, \ldots, y_n)$ starting at x, ending in $U_\varepsilon(v_i)$ and not approaching $\frac{1}{2}$ (except, maybe, for y_n) closer than $2C_6\varepsilon^\beta$. Then by Lemma 2.3 one can find $z^\omega \in I$ such that

$$\operatorname{dist}(y_k, f_\lambda^k z^\omega) \leqq C_6\varepsilon^{1-2\beta}, \qquad y_0 = x, \quad k = 0, \ldots, n-1 \tag{4.10}$$

and therefore

$$\operatorname{dist}(f_\lambda^k z^\omega, \tfrac{1}{2}) \geqq 2C_6\varepsilon^\beta - C_6\varepsilon^{1-2\beta} \geqq C_6\varepsilon^\beta \qquad \text{for all } k = 0, \ldots, n \tag{4.11}$$

provided $\beta < \frac{1}{3}$.

Consider all connected components Z_{ij} of the intersection $U_{C_6\varepsilon^{1-2\beta}}(x) \cap f_\lambda^{-n} U_\varepsilon(v_i)$ containing a point $z_{ij} \in Z_{ij}$ such that

$$\operatorname{dist}(f_\lambda^k z_{ij}, \tfrac{1}{2}) > \varepsilon^{2\beta} \qquad \text{for all } k = 0, \ldots, n-1. \tag{4.12}$$

It follows from (1.15) that if $z^\omega \in U_{C_6\varepsilon^{1-2\beta}}(x)$, $f_\lambda^n z^\omega \in U_{\varepsilon + C_6\varepsilon^{1-2\beta}}(v_i)$ and (4.11) holds then there exists z_{ij} constructed above satisfying (4.12) and

$$\operatorname{dist}(f_\lambda^k z_{ij}, f_\lambda^k z^\omega) \leqq \tfrac{1}{2}\varepsilon^{1-3\beta} \qquad \text{for all } k = 0, \ldots, n \tag{4.13}$$

provided $\varepsilon > 0$ is small enough. Hence by (4.10) and the argument preceding it one concludes that for any ε^α-pseudo-orbit $\omega = (x, y_1, \ldots, y_n)$ starting at x, ending in $U_\varepsilon(v_i)$, with $y_0 = x, y_1, \ldots, y_{n-1}$ staying outside of $U_{2C_6\varepsilon^\beta}(\frac{1}{2})$ there exists z_{ij} introduced above such that

$$\operatorname{dist}(f_\lambda^k z_{ij}, y_k) \leqq \varepsilon^{1-3\beta} \qquad \text{for all } k = 0, \ldots, n. \tag{4.14}$$

Therefore

$$J_2^\varepsilon(2C_6\varepsilon^\beta, \varepsilon^{1-\beta}, n, x, U_\varepsilon(v_i)) \leqq \sum_j J_3^\varepsilon(\varepsilon^{1-3\beta}, n, x; z_{ij}, U_\varepsilon(v_i)) \tag{4.15}$$

where

$$J_3^\varepsilon(\delta, n, x; z, \Gamma) = P_x\{x_k^\varepsilon \in U_\delta(f_\lambda^k z) \text{ for all } k = 0, \dots, n-1 \text{ and } x_n^\varepsilon \in \Gamma \cap U_\delta(f_\lambda^n z)\}$$

(4.16)

$$= \int_{U_\delta(f_\lambda z)} \cdots \int_{U_\delta(f_\lambda^{n-1} z)} \int_{U_\delta(f_\lambda^n z)\cap\Gamma} q_{f_\lambda x}^\varepsilon(y_1) q_{f_\lambda y_1}^\varepsilon(y_2) \cdots q_{f_\lambda y_{n-1}}^\varepsilon(y_n) dy_1 \cdots dy_n.$$

Let $\beta > 0$ be so small that

$$1 - 4\beta > \alpha_2 \quad \text{and} \quad \beta < \tfrac{1}{18} \tag{4.17}$$

and suppose

$$x \in [\varepsilon^{\alpha_2+\beta}, 1 - \varepsilon^{\alpha_2+\beta}]. \tag{4.18}$$

If $y_k \in U_{\varepsilon^{1-3\beta}}(f_\lambda^k z_{ij})$ and $y_{k+1} \in U_{\varepsilon^{1-3\beta}}(f_\lambda^k z_{ij})$ for $k = 0, 1, \dots, n-1$ then one can see from (4.12), (4.14), (4.17) and (4.18) that $\operatorname{dist}(y_k, 0) > \varepsilon^{5\beta}$ for all $k = 0, \dots, n$. Therefore

$$|f_\lambda y_k - y_{k+1}| < \varepsilon^{\alpha_2} \tag{4.19}$$

provided ε is small enough and f_λ has bounded second derivatives on any interval $(y_k, f_\lambda^k z_{ij})$. Then by (4.14),

$$|y_{k+1} - f_\lambda y_k - (y_{k+1} - f_\lambda^{k+1} z_{ij}) + f_\lambda'(f_\lambda^k z_{ij})(y_k - f_\lambda^k z_{ij})| \leqq C_{18}\varepsilon^{2-6\beta} \tag{4.20}$$

for some $C_{18} > 0$.

Since (4.19) is true one can employ (1.8) to obtain

$$q_{f_\lambda y_k}^\varepsilon(y_{k+1}) \leqq (1 + \varepsilon^{\alpha_2})\varepsilon^{-1} r_{f_\lambda y_k}\left(\frac{y_{k+1} - f_\lambda y_k}{\varepsilon}\right). \tag{4.21}$$

Set $\eta_l = y_l - f_\lambda^l z_{ij}$. Then by (4.20) and Assumption A(iii) it follows that

$$\left| r_{f_\lambda y_k}\left(\frac{y_{k+1} - f_\lambda y_k}{\varepsilon}\right) - r_{f_\lambda^{k+1} z_{ij}}\left(\frac{\eta_{k+1} - f_\lambda'(f_\lambda^k z_{ij})\eta_k}{\varepsilon}\right)\right| \leqq \varepsilon^{1-7\beta} \tag{4.22}$$

provided ε is small enough and

$$\text{either} \quad \varepsilon^{-1}(\eta_{k+1} - f_\lambda'(f_\lambda^k z_{ij})\eta) \in V^+_{f_\lambda^{k+1} z_{ij}} \quad \text{or} \quad \varepsilon^{-1}(y_{k+1} - f_\lambda y_k) \notin V^+_{f_\lambda y_k}. \tag{4.23}$$

If (4.23) is not satisfied then, by (4.20) and Assumption A(iv),

$$y_{k+1} \in \partial_{k+1}^\varepsilon \equiv \{\tilde{y} : \tilde{y} - f_\lambda y_k \in \partial V^+_{f_\lambda y_k}(\varepsilon^{1-7\beta})\} \tag{4.24}$$

provided ε is small enough and so one can drop the restriction (4.23) by modifying (4.22) in the following way:

$$\begin{aligned} r_{f_\lambda y_k}(\varepsilon^{-1}(y_{k+1} - f_\lambda y_k)) \leqq\ & r_{f_\lambda^{k+1} z_{ij}}(\varepsilon^{-1}(\eta_{k+1} - f_\lambda'(f_\lambda^k z_{ij})\eta_k) + \varepsilon^{1-7\beta} \\ & + \chi_{\partial_{k+1}^\varepsilon}(y_{k+1}) r_{f_\lambda y_k}(\varepsilon^{-1}(y_{k+1} - f_\lambda y_k)) \end{aligned} \tag{4.25}$$

where, as usual, $\chi_\Gamma(y)=1$ if $y\in\Gamma$ and $\chi_\Gamma(y)=0$ otherwise.

From Assumption A(iv) it follows that

$$(4.26)\qquad \int_{\partial^\varepsilon_{k+1}\cap U_{\varepsilon^{1-3\beta}}(f_\lambda^{k+1}z_{ij})} \varepsilon^{-1} r_{f_\lambda y_k}(\varepsilon^{-1}(y_{k+1}-f_\lambda y_k))dy_{k+1} \leqq 2C_2\varepsilon^{1-7\beta}$$

provided $y_k\in U_{\varepsilon^{1-3\beta}}(f_\lambda^{k+1}z)$ and ε is small enough.

Now from (3.6)–(3.8), (3.10), (4.16), (4.25), (4.26) substituting $\xi=x-z_{ij}$ and $\eta_{k+1}=y_{k+1}-f_\lambda^{k+1}z_{ij}$ one obtains

$$J_3^\varepsilon(\varepsilon^{1-3\beta}, n, x; z_{ij}, U_\varepsilon(v_i))$$
$$(4.27)\quad \leqq C_{19}\varepsilon^{-1}B_n^{-1/2}(z_{ij})|(f_\lambda^n)'(z_{ij})|^{-1}(\varepsilon^{1-14\beta}+\exp\{-\alpha B_n^{-1/2}(z_{ij})\varepsilon^{-1}|\xi+\beta_{z_{ij}}(n)|\})$$

(cf. the formula (5.35) of [5]) for some $C_{19}>0$.

Next we shall prove that for any $z_{i_1j_1}, z_{i_2j_2}\in U_{C_6\varepsilon^{1-2\beta}}(x)$ satisfying (4.12) one has

$$(4.28)\qquad C_{20}^{-1}\leqq\frac{B_n(z_{i_1j_1})}{B_n(z_{i_2j_2})}\leqq C_{20}\quad\text{and}\quad|\beta_{z_{i_1j_1}}(n)-\beta_{z_{i_2j_2}}(n)|\leqq C_{20}\varepsilon^{5\beta}$$

with $C_{20}>0$ independent of i_1, j_1, i_2, j_2.

Moreover we shall show that there exists an integer N depending on ε such that

$$(4.29)\qquad C_{21}^{-1}\leqq\frac{|(f_\lambda^k)'(z_{i_1j_1})|}{|(f_\lambda^k)'(z_{i_2j_2})|}\leqq(1+C_{21}\varepsilon^{1-12\beta})\qquad\text{for all } k=1,\dots,N$$

and

$$(4.30)\qquad |(f_\lambda^l)'(z_{ij})|^{-1}\leqq C_{21}\varepsilon^{1-6\beta}\qquad\text{for all } l=N+1,\dots$$

where $C_{21}>0$ is independent of i_1, i_2, j_1, j_2, i, j.

Indeed, let

$$(4.31)\qquad N=\max\{k:\operatorname{dist}(f_\lambda^k z_{i_1j_1}, f_\lambda^k z_{i_2j_2})\leqq\varepsilon^{1-10\beta}\}.$$

Then from (4.12) and Lemma 2.2 one can see that (4.29) holds with some $C_{21}>0$.

Since $\operatorname{dist}(z_{i_1j_1}, z_{i_2j_2})\leqq 2C_6\varepsilon^{1-2\beta}$ and $\operatorname{dist}(f_\lambda^{N+1}z_{i_1j_1}, f_\lambda^{N+1}z_{i_2j_2})\geqq\varepsilon^{1-10\beta}$, then by (4.29)

$$(4.32)\qquad |(f_\lambda^{N+1})'(z_{i_1j_1})|\geqq C_{22}^{-1}\varepsilon^{-8\beta}$$

with $C_{22}>0$ independent of ε, i_1 and j_1. Using again (4.12) and Lemma 2.2 one can see from here that

$$(4.33)\qquad |(f_\lambda^{N+k})'(z_{i_1j_1})|^{-1}\leqq C_{23}\varepsilon^{1-6\beta}$$

for some $C_{23}>0$ and all $k=1,\dots$.

This proves (4.29)–(4.30) and so (4.28) is also true since $\beta<\frac{1}{18}$, $\sup_y|f_\lambda'(y)|<4\lambda$, $n\leqq(\ln\varepsilon)^4$ and $b(y)$ in the definition (3.11) of $\beta_x(n)$ is bounded and Lipshitz continuous by the conditions (i)–(iv) of §1.

Hence setting $B_n = B_n(z_{i_1 j_1})$ and $\beta(n) = \beta_{z_{i_1 j_1}}(n)$ we can rewrite (4.27) as follows:

$$J_3^\varepsilon(\varepsilon^{1-3\beta}, n, x; z_{ij}, U_\varepsilon(v_i))$$
$$\leqq C_{24}\varepsilon^{-1}B_n^{-1/2}|(f_\lambda^n)'(z_{ij})|^{-1}\left(\varepsilon^{1-14\beta} + \exp\left\{-\alpha_4 B_n^{-1/2}\left|\frac{x - z_{ij}}{\varepsilon} + \beta(n)\right|\right\}\right) \tag{4.34}$$

for some C_{24}, $\alpha_4 > 0$.

Let

$$\mathscr{F}_k = \left\{z_{ij}:\ k \leqq B_n^{-1/2}\left|\frac{x - z_{ij}}{\varepsilon} + \beta(n)\right| < k+1\right\}.$$

Then, by Lemma 2.5,

$$B_n^{-1/2}\varepsilon^{-1}\sum_{z_{ij}\in\mathscr{F}_k}|(f_\lambda^n)'(z_{ij})|^{-1} \leqq 2Z_\delta. \tag{4.35}$$

Since $|(f_\lambda^l)'(z)| \leqq (4\lambda)^l$ for any l and z then, by (4.13), one has $B_n^{-1/2} \leqq \lambda^{-1/2}$. Besides, from (2.12) and Lemma 2.2 it follows easily that $|\beta(n)| \leqq \text{const}\,\varepsilon^{-2\beta}$.

But we consider z_{ij} satisfying $|x - z_{ij}| \leqq C_6\varepsilon^{1-2\beta}$, and so the sets $\mathscr{F}_k$ should be taken into account only when $0 \leqq k \leqq \varepsilon^{-3\beta}$ provided ε is small enough. Here we have used $|x - z_{ij}|$ instead of $\text{dist}(x, z_{ij})$ because this is the same in view of (4.17), (4.18) and $|x - z_{ij}| \leqq C_6\varepsilon^{1-2\beta}$, provided ε is small enough.

Now from (4.15), (4.17), (4.34) and (4.35) one obtains

$$J_2^\varepsilon(2C_6\varepsilon^\beta, \varepsilon^{1-\beta}, n, x, U_\varepsilon(v_i)) \leqq 2C_{24}Z_\delta\varepsilon \sum_{0\leqq k\leqq\varepsilon^{-3\beta}}(\varepsilon^{1-14\beta} + e^{-\alpha_4 k}) \leqq C_{25}Z_\delta\varepsilon \tag{4.36}$$

provided ε is small enough, where $C_{25} > 0$ is independent of ε, δ, i.

Finally, by (4.7), (4.8) and (4.17)

$$J_2^\varepsilon(2C_6\varepsilon^\beta, \varepsilon^{1-\beta}, n, x, Q) \leqq C_{25}Z_\delta \text{ mes } Q. \tag{4.37}$$

Since $n \leqq (\ln\varepsilon)^4$ and (4.17) holds, then (4.2) follows from (4.5) and (4.37) with $\gamma_0 = \beta$ and $D_\delta = C_{25}Z_\delta$ proving Lemma 4.1.

Since the derivative of f_λ is not continuous at 0 we have to treat this point separately.

Lemma 4.2. *There exists $C_{26} > 0$ such that if $(\ln\varepsilon)^4 \geqq n \geqq (\ln\varepsilon)^2$ and $\varepsilon > 0$ is small enough, then for any $\delta \leqq \frac{1}{8}$ and $x \in U_\delta(0)$ one has*

$$J_4^\varepsilon(\delta, n, x) \equiv P\left\{\max_{0\leqq k\leqq n}\text{dist}(x_k^\varepsilon, 0) \leqq \delta\right\} \leqq C_{26}(2\lambda)^{-(n-(\ln\varepsilon)^2/2)}. \tag{4.38}$$

Proof. By the Chapman–Kolmogorov formula

$$J_4^\varepsilon(\delta, n, x) = \int_{U_\delta(0)}\cdots\int_{U_\delta(0)} q_{f_\lambda x}^\varepsilon(y_1)q_{f_\lambda y_1}^\varepsilon(y_2)\cdots q_{f_\lambda y_{n-1}}^\varepsilon(y_n)\,dy_1\cdots dy_n$$
$$= J_5^\varepsilon(\delta, n, x) + R^\varepsilon(\delta, n, x) \tag{4.39}$$

where

$$J_5^\varepsilon(\delta, n, x) \equiv \int_{U_\delta(0)\cap U_{\varepsilon^{1-\beta}}(f_\lambda x)} \int_{U_\delta(0)\cap U_{\varepsilon^{1-\beta}}(f_\lambda y_1)} \cdots \int_{U_\delta(0)\cap U_{\varepsilon^{1-\beta}}(f_\lambda y_{n-1})} \tag{4.40}$$
$$\times\, q^\varepsilon_{f_\lambda x}(y_1) q^\varepsilon_{f_\lambda y_1}(y_2) \cdots q^\varepsilon_{f_\lambda y_{n-1}}(y_n) dy_1 \cdots dy_n,$$

$\beta > 0$ satisfies (4.17) and $R^\varepsilon(\delta, n, x)$ is defined by (4.39) and (4.40).

By (1.1), (1.8), (1.9) and (1.11), in the same way as in (4.5), one has

$$R^\varepsilon(\delta, n, x) \leqq C_{27} n \exp(-\alpha_5 \varepsilon^{-\beta}) \tag{4.41}$$

for some C_{27}, $\alpha_5 > 0$ independent of ε and n.

The integration in (4.40) is over $\varepsilon^{1-\beta}$-pseudo-orbits $\omega = (x, y_1, \ldots, y_n)$ starting at x and staying in the δ-neighbourhood of the point 0. Since $\delta \leqq \frac{1}{8}$ then $\operatorname{dist}(y_i, \frac{1}{2}) \geqq \frac{3}{8}$ and by Lemma 2.3 one concludes that there exists a point z^ω such that

$$\operatorname{dist}(f_\lambda^i z^\omega, y_i) \leqq C_{28}\varepsilon^{1-\beta}, \qquad i = 0, 1, \ldots, n, \quad y_0 = x \tag{4.42}$$

where $C_{28} > 0$ is independent of $x, y_1, \ldots, y_n$ and ε. Then

$$f_\lambda^i z^\omega \in U_{\delta + C_{28}\varepsilon^{1-\beta}}(0) \qquad \text{for all } i = 0, 1, \ldots, n. \tag{4.43}$$

If ε is small enough then $\delta + C_{28}\varepsilon^{1-\beta} < \frac{1}{7}$, and since $|f_\lambda'(x)| \geqq \frac{10}{7} > 1$ for any $x \in U_{1/7}(0)$ when $\lambda \geqq \frac{1}{2}$, one obtains from (4.43) that

$$f_\lambda^i z^\omega \in U_\varepsilon(0) \quad \text{for all } i \leqq n - \tfrac{1}{4}(\ln \varepsilon)^2 \quad \text{provided } \varepsilon \text{ is small enough.} \tag{4.44}$$

Hence by (4.39), (4.40), (4.42) and (4.44),

$$J_5^\varepsilon(\delta, n, x) \leqq J_4^\varepsilon(\varepsilon^{1-2\beta}, n - [\tfrac{1}{4}(\ln \varepsilon)^2] - 1, x) \tag{4.45}$$

and either $x \in U_{\varepsilon^{1-2\beta}}(0)$ or $J_5^\varepsilon(\delta, n, x) = 0$, where ε is supposed to be small enough and $[\cdot]$ denotes the integral part.

Now (1.8), (1.9), (4.17) and (4.39) yield, for $x \in U_{\varepsilon^{1-2\beta}}(0)$ and any k,

$$J_4^\varepsilon(\varepsilon^{1-2\beta}, k, x) \leqq (1 + \varepsilon^{\alpha_2})^k \int_{U_{\varepsilon^{1-2\beta}}(0)} \cdots \int_{U_{\varepsilon^{1-2\beta}}(0)} \varepsilon^{-1} r_x\left(\frac{\sigma(x, y_1)}{\varepsilon}\right) \varepsilon^{-1} r_{f_\lambda y_1}\left(\frac{\sigma(f_\lambda y_1, y_2)}{\varepsilon}\right) \tag{4.46}$$
$$\cdots \varepsilon^{-1} r_{f_\lambda y_{k-1}}\left(\frac{\sigma(f_\lambda y_{k-1}, y_k)}{\varepsilon}\right) dy_1 \cdots dy_k$$

where

$$\sigma(y, z) \equiv \begin{cases} z - y & \text{if } |z - y| \leqq \varepsilon^{\alpha_2}, \\ z - y + 1 & \text{if } |z - y + 1| \leqq \varepsilon^{\alpha_2}, \\ z - y - 1 & \text{if } |z - y - 1| \leqq \varepsilon^{\alpha_2}. \end{cases} \tag{4.47}$$

It is easy to see that

$$|\sigma(fy_i, y_{i+1}) - \sigma(0, y_{i+1}) + 4\lambda\,|\sigma(0, y_i)|\,| \leqq C_{29}\varepsilon^{2-4\beta} \tag{4.48}$$

since $y_i \in U_{\varepsilon^{1-2\beta}}(0)$ for all $i = 0, 1, \dots, n$.

Then in the same way as in (4.25) one has

$$\begin{aligned} r_{f_\lambda y_i}\left(\frac{\sigma(f_\lambda y_i, y_{i+1})}{\varepsilon}\right) &\leqq r_0(\varepsilon^{-1}(\eta_{i+1} - 4\lambda\,|\eta_i|)) + \varepsilon^{1-5\beta} \\ &\quad + \chi_{\tilde\delta^\varepsilon_{i+1}}(y_{i+1}) r_{f_\lambda y_i}\left(\frac{\sigma(f_\lambda y_i, y_{i+1})}{\varepsilon}\right) \end{aligned} \tag{4.49}$$

where $\eta_i = \sigma(0, y_i)$ and

$$\tilde\delta^\varepsilon_{i+1} \equiv \{\tilde y \colon \sigma(f_\lambda y_i, \tilde y) \in \partial V^+_{f_\lambda y_i}(\varepsilon^{1-5\beta})\}. \tag{4.50}$$

Next, similarly to (4.27), from Assumption A(iv), (3.25)–(3.27) and (4.46)–(4.50) it follows that

$$J_4^\varepsilon(\varepsilon^{1-2\beta}, k, x) \leqq C_{30}(1 + \varepsilon^{\alpha_2})^k \varepsilon^{-1}(2\lambda)^{-k} \tag{4.51}$$

for some $C_{30} > 0$ independent of ε, k and x.

Finally (4.39)–(4.41) together with (4.45) and (4.51) yield (4.38) for ε small enough, proving Lemma 4.2.

From Lemma 4.2 it follows that we can drop the restriction $x \in [\varepsilon^{\alpha_2+\gamma_0}, 1 - \varepsilon^{\alpha_2+\gamma_0}]$ in Lemma 4.1.

Corollary 4.1. *For any $x \in I$ and an interval $Q \subset I$ satisfying* (2.31) *with some $\delta > 0$ one has*

$$J_1^\varepsilon(\varepsilon^\gamma, n, x, Q) \leqq 2D_\delta \operatorname{mes} Q \tag{4.52}$$

provided $(\ln \varepsilon)^4 \geqq n \geqq 2(\ln \varepsilon)^2 + 1$, $\gamma \leqq \gamma_0$ *and ε is small enough, where D_δ is the same as in* (4.2).

Proof. Since for some $C_{31} > 0$ independent of ε, y and z

$$q_y^\varepsilon(z) \leqq C_{31}\varepsilon^{-1} \qquad \text{for all } y, z \text{ and } \varepsilon > 0, \tag{4.53}$$

then

$$J_1^\varepsilon(\varepsilon^\gamma, k, x, Q) \leqq C_{31}\varepsilon^{-1} \operatorname{mes} Q \qquad \text{for any } k. \tag{4.54}$$

Define

$$\tau_\varepsilon = \min\{k \colon x_x^\varepsilon \notin U_{\varepsilon^{\alpha_2}}(0)\}. \tag{4.55}$$

For $x \notin U_{\varepsilon^{\alpha_2}}(0)$ the inequality (4.52) is proved in Lemma 4.1. So assume that $x \in U_{\varepsilon^{\alpha_2}}(0)$. Since τ_ε is the Markov time, then by the strong Markov property of the process x_k^ε it follows from (4.2), (4.38) and (4.54) that

$$
\begin{aligned}
J_1^\varepsilon(\varepsilon^\gamma, n, x, Q) &= \sum_{1\leqq k\leqq n} EP_x\left\{\max_{0\leqq i\leqq k-1} \operatorname{dist}(x_i^\varepsilon, 0)\leqq \varepsilon^{\alpha_2}, \tau_\varepsilon = k\right\} J_1^\varepsilon(\varepsilon^\gamma, n-k, x_k^\varepsilon, Q)\\
&\leqq \sum_{1\leqq k\leqq (\ln\varepsilon)^2} P_x\{\tau_\varepsilon = k\} J_1^\varepsilon(\varepsilon^\gamma, n-k, x_k^\varepsilon, Q)\\
&\quad + C_{31}\varepsilon^{-1}\operatorname{mes} Q \sum_{1+(\ln\varepsilon)^2\leqq k\leqq (\ln\varepsilon)^4} P_x\left\{\max_{0\leqq i\leqq k-1}\operatorname{dist}(x_i^\varepsilon,0)\leqq \varepsilon^{\alpha_2}\right\}\\
&\leqq D_\delta \operatorname{mes} Q + C_{26}C_{31}\varepsilon^{-1}(\ln\varepsilon)^8(2\lambda)^{-(\ln\varepsilon)^2/2}\operatorname{mes} Q,
\end{aligned}
\tag{4.56}
$$

since $(\ln\varepsilon)^4 \geqq n \geqq 2(\ln\varepsilon)^2+1$, and so when $k\leqq(\ln\varepsilon)^2$ then $n-k\geqq(\ln\varepsilon)^2+1$, where E denotes the expectation. Recall that $\lambda>\frac{1}{2}$, and so (4.52) follows from (4.56) provided ε is small enough, proving Corollary 4.1.

Next we shall take care of paths of the Markov chain x_k^ε which sometimes approach the point $\frac{1}{2}$.

Define

$$
\begin{aligned}
&P^\varepsilon_{i_1,i_2,\ldots,i_k}(\gamma, n, x, Q)\\
&\equiv P_x\{x_j^\varepsilon \in U_{\varepsilon^\gamma}(\tfrac{1}{2}) \text{ for } j = i_1, i_2, \ldots, i_k < n,\ x_j^\varepsilon \notin U_{\varepsilon^\beta}(\tfrac{1}{2}) \text{ if } j\neq i_l\\
&\qquad \text{for some } l \text{ and } x_n^\varepsilon \in Q\}.
\end{aligned}
\tag{4.57}
$$

Lemma 4.3. *There exists γ_1, $C_{32}>0$ such that for any $x\in I$ and an interval $Q\subset I$ one has*

$$
P^\varepsilon_{i_1,\ldots,i_k}(\gamma, n, x, Q)\leqq C_{32}\varepsilon^{(k-1)\gamma-2}\operatorname{mes} Q \tag{4.58}
$$

provided $\gamma\leqq\gamma_1$, $n\leqq(\ln\varepsilon)^4$ and $\varepsilon>0$ is small enough.

Proof. Let $g^\varepsilon_{i_1,\ldots,i_k}(n,x,y)$ be the density of $P^\varepsilon_{i_1,\ldots,i_k}(n,x,Q)$, i.e.

$$
P^\varepsilon_{i_1,\ldots,i_k}(n,x,Q) = \int_Q g^\varepsilon_{i_1,\ldots,i_k}(n,x,y)\,dy, \tag{4.59}
$$

then by the Chapman–Kolmogorov formula

$$
g^\varepsilon_{i_1,\ldots,i_l}(i_{l+1},x,y) = \int_{U_{\varepsilon^\gamma}(1/2)} g^\varepsilon_{i_1,\ldots,i_{l-1}}(i_l,x,z)\,g^\varepsilon_{i_l}(i_{l+1}-i_l,z,y)\,dz \tag{4.60}
$$

for any $l=1,\ldots,k$ where we put $i_{k+1}\equiv n$.

Denote also

$$
g_0^\varepsilon(m,y,z)\equiv \int_{I\setminus U_{\varepsilon^\gamma}(1/2)}\cdots\int_{I\setminus U_{\varepsilon^\gamma}(1/2)} q^\varepsilon_{f_\lambda y}(y_1)g^\varepsilon_{f_\lambda y_1}(y_2)\cdots q^\varepsilon_{f_\lambda y_{m-2}}(y_{m-1})q^\varepsilon_{f_\lambda y_{m-1}}(z)\,dy_1\cdots dy_{m-1}, \tag{4.61}
$$

then by (1.1), (1.8), (1.9) and (1.11),

$$g_0^\varepsilon(i_1, x, z) \leqq 2C_0\varepsilon^{-1} \equiv A_0^\varepsilon \tag{4.62}$$

provided $\varepsilon > 0$ is small enough.

Suppose that

$$g^\varepsilon_{i_1,\dots,i_{l-1}}(i_l, x, z) \leqq A^\varepsilon_{l-1} \qquad \text{for any } z \in U_{\varepsilon^\gamma}(\tfrac{1}{2}). \tag{4.63}$$

Then by (4.60),

$$\begin{aligned} g^\varepsilon_{i_1,\dots,i_l}(i_{l+1}, x, y) &\leqq A^\varepsilon_{l-1} \int_{U_{\varepsilon^\gamma}(1/2)} g^\varepsilon_{i_l}(i_{l+1} - i_l, z, y)dz \\ &= A^\varepsilon_{l-1} \int_{U_{\varepsilon^\gamma}(1/2)} \int_I q^\varepsilon_{f_\lambda z}(v) g^\varepsilon_0(i_{l+1} - i_l - 1, v, y)dz\, dv \\ &= A^\varepsilon_{l-1} \sum_{0 \leqq j \leqq m} \int_{U^{(j+1)}(1/2) \setminus U^{(j)}(1/2)} \int_I q^\varepsilon_{f_\lambda z}(v) g^\varepsilon_0(i_{l+1} - i_l - 1, v, y)dz\, dv \end{aligned} \tag{4.64}$$

where $U^{(j)}(\tfrac{1}{2}) = U_{\varepsilon^{\beta_j}}(\tfrac{1}{2})$ for $j = 1, \dots, m$, $U^{(0)}(\tfrac{1}{2}) = \varnothing$, $\beta_{m+1} = \gamma$,

$$1 > 2\beta_{j+1} > \beta_j > \beta_{j+1} \geqq \gamma \qquad \text{for all } j = 1, \dots, m \quad \text{and} \quad 2\beta_2 > 1 - \beta_1. \tag{4.65}$$

If $z \in U^{(j)}(\tfrac{1}{2})$ then $f_\lambda z \in U_{4\lambda_\varepsilon^{2\beta_j}}(f_\lambda(\tfrac{1}{2}))$ and so by (1.1), (1.8), (1.9) and (1.11)

$$q^\varepsilon_{f_\lambda z}(v) \leqq C_{40} \exp(-\alpha_6 \varepsilon^{2\beta_j - 1}) \qquad \text{if } z \in U^{(j)}(\tfrac{1}{2}) \quad \text{and} \quad v \notin U_{5\lambda_\varepsilon^{2\beta_j}}(f_\lambda(\tfrac{1}{2})) \tag{4.66}$$

for some C_{40}, $\alpha_6 > 0$. Since $2\beta_j < 1$ one obtains from here that

$$\begin{aligned} &\int_{U^{(j)}(1/2) \setminus U^{(j-1)}(1/2)} \int_I q^\varepsilon_{f_\lambda z}(v) g^\varepsilon_0(i_{l+1} - i_l - 1, v, y)dz\, dv \\ &\leqq \int_{U^{(j)}(1/2) \setminus U^{(j-1)}(1/2)} \int_{U_{5\lambda_\varepsilon^{2\beta_j}}(f_\lambda(1/2))} q^\varepsilon_{f_\lambda z}(v) g^\varepsilon_0(i_{l+1} - i_l - 1, v, y)dz\, dv + \exp(-\alpha_7 \varepsilon^{2\beta_j - 1}) \end{aligned} \tag{4.67}$$

for some $\alpha_7 > 0$ provided ε is small enough.

It is easy to see that

$$\int_{U^{(1)}(1/2)} q^\varepsilon_{f_\lambda z}(v)dz \leqq 2C_0\varepsilon^{\beta_1 - 1} \tag{4.68}$$

where C_0 is the same as in (1.1) and

$$\int_{U^{(j+1)}(1/2) \setminus U^{(j)}(1/2)} q^\varepsilon_{f_\lambda z}(v)dz = \int_{f_\lambda(U^{(j+1)}(1/2) \setminus U^{(j)}(1/2))} q^\varepsilon_w(v) \sum_{z \in f_\lambda^{-1} w} |f'_\lambda(z)|^{-1} dw \leqq C_{34}\varepsilon^{-\beta_j}, \tag{4.69}$$

for some $C_{34}>0$ since

$$\int_I q_w^\varepsilon(v)dw \leqq 1+2\varepsilon^{\alpha_2}. \tag{4.70}$$

Indeed, by Assumption A(i)–(ii), (1.8) and (1.11), if $\tilde{\alpha}=\frac{1}{2}\min(\alpha_1,\alpha_2)$ then

$$\begin{aligned}\int_I q_w^\varepsilon(v)dw &\leqq \int_{U_{\varepsilon^{1-\beta}}(v)} q_w^\varepsilon(v)dw+\exp(-\tilde{\alpha}\varepsilon^{-\beta})\\ &\leqq (1+\varepsilon^{\alpha_3})\varepsilon^{-1}\int_I r_v\left(\frac{w-v}{\varepsilon}\right)dw+\exp(-\tilde{\alpha}\varepsilon^{-\beta})\\ &\leqq (1+2\varepsilon^{\alpha_2})\end{aligned} \tag{4.71}$$

provided $0<\beta<\frac{1}{2}$ and ε is small enough.

Now collecting (4.64), (4.67)–(4.69) one obtains

$$\begin{aligned} g^\varepsilon_{i_1,\dots,i_l}(i_{l+1},x,y) \leqq\, & 2A^\varepsilon_{l-1}C_0\varepsilon^{\beta_l-1}\int_{U_{5\lambda\varepsilon^{2\beta_1}}(f_\lambda(1/2))} g_0^\varepsilon(i_{l+1}-i_l-1,v,y)dv\\ &+A^\varepsilon_{l-1}C_{41}\sum_{2\leqq j\leqq m}\varepsilon^{-\beta_{j-1}}\int_{U_{5\lambda\varepsilon^{2\beta_j}}(f_\lambda(1/2))} g_0^\varepsilon(i_{l+1}-i_l-1,v,y)dv.\end{aligned} \tag{4.72}$$

Next, in the same way as in (4.5) it follows from (1.1), (1.8), (1.9) and (1.11) that

$$g_0^\varepsilon(i_{l+1}-i_l-1,v,y)\leqq \tilde{g}_0^\varepsilon(i_{l+1}-i_l-1,v,y)+e^{-\alpha_8\varepsilon^\gamma} \tag{4.73}$$

for some $\alpha_8>0$ provided ε is small enough, where $i_{l+1}-i_l\leqq n\leqq(\ln\varepsilon)^4$,

$$\begin{aligned}&\tilde{g}_0^\varepsilon(m,v,y)\\ &\equiv \int_{U_{\varepsilon^{1-\gamma}}(f_\lambda v)\cap(I\setminus U_{\varepsilon^\gamma}(1/2))}\cdots\int_{U_{\varepsilon^{1-\gamma}}(f_\lambda y_{m-3})\cap(I\setminus U_{\varepsilon^\gamma}(1/2))}\int_{U_{\varepsilon^{1-\gamma}}(f_\lambda y_{m-2})\cap U_{\varepsilon^{1-\gamma}}(f_\lambda^{-1}y)\cap(I\setminus U_{\varepsilon^\gamma}(1/2))}\\ &\qquad\times q^\varepsilon_{f_\lambda z}(z_1)q^\varepsilon_{f_\lambda z_1}(z_2)\cdots q^\varepsilon_{f_\lambda z_{m-2}}(z_{m-1})q^\varepsilon_{f_\lambda z_{m-1}}(y)dz_1\cdots dz_{m-1}.\end{aligned} \tag{4.74}$$

The integration in (4.74) is over $\varepsilon^{1-\gamma}$-pseudo-orbits $\omega=(v,z_1,\dots,z_{m-1},y)$ starting at v, ending at y and not approaching $\frac{1}{2}$ (except for the last point y) closer than ε^γ. Then by Lemma 2.3 one can find $z^\omega\in I$ such that

$$\operatorname{dist}(z_i,f_\lambda^i z^\omega)\leqq\varepsilon^{1-3\gamma},\qquad i=0,\dots,m;\quad z_0=v,\quad z_m=y \tag{4.75}$$

provided ε is small enough.

Take one point z^{ij} in each connected component Z^{ij} of the intersection

$$f_\lambda^{-(i_{l+1}-i_l-1)}U_{\varepsilon^{1-3\gamma}}(y)\cap U_{6\lambda\varepsilon^{2\beta_j}}(f_\lambda(\tfrac{1}{2}))$$

where $y \in U_{\varepsilon^\gamma}(\frac{1}{2})$, then it follows from above that

$$\int_{U_{5\lambda\varepsilon^{2\beta_j}}(f_\lambda(1/2))} \tilde{g}_0^\varepsilon(i_{l+1}-i_l-1, v, y)dv \leqq \sum_i \int_{U_{\varepsilon^{1-4\gamma}}(z^{ij})} h^\varepsilon(z^{ij}; i_{l+1}-i_l-1, v, y)dv$$

(4.76)
$$\leqq 2\varepsilon^{1-4\gamma}\sum_i \sup_{v\in U_{\varepsilon^{1-4\gamma}}(z^{ij})} h^\varepsilon(z^{ij}; i_{l+1}-i_l-1, v, y)$$

provided ε is small enough where

$$h^\varepsilon(z, m, v, w) \equiv \int_{U_{\varepsilon^{1-3\gamma}}(f_\lambda z)} \cdots \int_{U_{\varepsilon^{1-3\gamma}}(f_\lambda^m z)} q^\varepsilon_{f_\lambda v}(y_1)q^\varepsilon_{f_\lambda y_1}(y_2)\cdots q^\varepsilon_{f_\lambda y_{m-1}}(w)dy_1\cdots dy_{m-1}.$$

(4.77)

First, suppose that $\lambda < 1$, and so if ε is small enough then

(4.78) $$U_{6\lambda\varepsilon^{2\beta_j}+\varepsilon^{1-4\gamma}}(f_\lambda(\tfrac{1}{2})) \subset [\varepsilon^\alpha, 1-\varepsilon^\alpha].$$

Using (4.21), (4.25) and (4.26) in the same way as in (4.27) and taking into account that (3.8) divided by mes Ψ is the inequality for densities, one concludes from (4.77) that

(4.79) $$\sup_{v\in U_{\varepsilon^{1-4\gamma}}(z^{ij})} h^\varepsilon(z^{ij}; i_{l+1}-i_l-1, v, y) \leqq C_{36}\varepsilon^{-1}|(f_\lambda^{i_{l+1}-i_l-1})'(z^{ij})|^{-1}$$

for some $C_{36} > 0$.

Since $z^{ij} \in U_{6\lambda\varepsilon^{2\beta_j}}(f_\lambda(\frac{1}{2}))$ and $f_\lambda^{i_{l+1}-i_l-1}z^{ij} \in U_{\varepsilon^\gamma+\varepsilon^{1-3\gamma}}(\frac{1}{2})$, then using Lemma 2.2 one can see that

(4.80) $$|(f_\lambda^{i_{l+1}-i_l-1})'(z^{ij})|^{-1} \leqq C_{37}\varepsilon^{2\beta_j}$$

for some $C_{37} > 0$ independent of ε, i and j.

Hence by Lemma 2.5 it follows that

(4.81) $$\sum_i |(f_\lambda^{i_{l+1}-i_l-1})'(z^{ij})|^{-1} \leqq C_{38}\varepsilon^{2\beta_j}.$$

Now by (4.72)–(4.74), (4.76), (4.79) and (4.81) it follows that

(4.82) $$g^\varepsilon_{i_1,\dots,i_l}(i_{l+1}, x, y) \leqq 2C_{36}C_{38}C_0A^\varepsilon_{l-1}\varepsilon^{-4\gamma}\left(3\varepsilon^{3\beta_1-1} + \sum_{2\leqq j\leqq m}\varepsilon^{2\beta_j-\beta_{j-1}}\right) \leqq A^\varepsilon_{l-1}\varepsilon^\gamma$$

provided ε is small enough and γ is chosen to satisfy

(4.83) $$0 < 5\gamma < \min(3\beta_1 - 1, 2\beta_j - \beta_{j-1})$$

which is possible since by (4.65) the right hand side of (4.83) is positive. Thus by (4.62), (4.63) and (4.82),

$$g^{\varepsilon}_{i_1,\ldots,i_{k-1}}(i_k, x, z) \leqq 2C_0\varepsilon^{-1}\varepsilon^{(k-1)\gamma} \qquad \text{for any } z \in U_{\varepsilon^\gamma}(\tfrac{1}{2}) \tag{4.84}$$

and so

$$\begin{aligned} g^{\varepsilon}_{i_1,\ldots,i_k}(n, x, y) &\leqq \int_{U_{\varepsilon^\gamma}(1/2)} g^{\varepsilon}_{i_1,\ldots,i_{k-1}}(i_k, x, z) g^{\varepsilon}_{i_k}(n - i_k, z, y)dz \\ &\leqq (2C_0)^2\varepsilon^{-2}\varepsilon^{(k-1)\gamma}; \end{aligned} \tag{4.85}$$

this together with (4.59) proves (4.58) under the condition $\lambda < 1$.

If $\lambda = 1$ then $f_\lambda(\frac{1}{2}) = 1$ and we have to take care about the endpoints. Let

$$\tau^{ij} = \min\{l\colon f^l_\lambda z^{ij} \notin U_{\varepsilon^{2\beta_j}}(f_\lambda(\tfrac{1}{2}))\} \qquad \text{if } j > 1$$

and

$$\tau^{i1} = \min\{l\colon f^l_\lambda z^{i1} \notin U_{\varepsilon^{2\beta_2}}(f_\lambda(\tfrac{1}{2}))\}.$$

Considering the integral $h^\varepsilon(z^{ij}, i_{l+1} - i_l - 1, v, y)$ defined by (4.77) we shall estimate $q^{\varepsilon}_{f_\lambda y_l}(y_{l+1})$ for $l = 0, \ldots, \tau^{ij} - 1$ by means of (1.9) and (4.49), and for $l = \tau^{ij}, \ldots, i_{l+1} - i_l - 2$ by means of (1.8) and (4.25). Now employing Lemma 3.5 one can see that

$$h^\varepsilon(z^{ij}; i_{l+1} - i_l - 1, v, y) \leqq C_{39}\varepsilon^{-1}(2\lambda)^{-\tau^{ij}} |(f_\lambda^{i_{l+1}-i_l-\tau^{ij}-1})'(f_\lambda^{\tau^{ij}} z^{ij})|^{-1} \tag{4.86}$$

for some $C_{39} > 0$.

Since $f_\lambda^{\tau^{ij}} z^{ij} \in U_{6\lambda\varepsilon^{2\beta_j}}$ if $j > 1$, $f_\lambda^{\tau^{i1}} z^{i1} \in U_{6\lambda\varepsilon^{2\beta_2}}$ and $f_\lambda^{i_{l+1}-i_l-1} z^{ij} \in U_{\varepsilon^\gamma + \varepsilon^{1-3\gamma}}(\frac{1}{2})$, then for some $C_{40} > 0$

$$|(f_\lambda^{i_{l+1}-i_l-\tau^{ij}-1})'(f_\lambda^{\tau^{ij}} z^{ij})|^{-1} \leqq C_{40}\varepsilon^{2\beta_j} \qquad \text{if } j > 1, \tag{4.87}$$

and if $j = 1$ then the right hand side of (4.87) should be replaced by $C_{40}\varepsilon^{2\beta_2}$. By Lemma 2.5 one can see from above that for some $C_{41} > 0$

$$\sum_{i, \tau^{ij} = k} |(f_\lambda^{i_{l+1}-i_l-k-1})'(f_z^k z^{ij})|^{-1} \leqq C_{41}\varepsilon^{2\beta_j} \qquad \text{if } j > 1, \tag{4.88}$$

and if $j = 1$ then we replace $C_{41}\varepsilon^{2\beta_j}$ by $C_{41}\varepsilon^{2\beta_2}$.

Finally, by (4.72)–(4.74), (4.76), (4.86) and (4.88) it follows that

$$g^{\varepsilon}_{i_1,\ldots,i_l}(i_{l+1}, x, y) \leqq C_{42}\varepsilon^{-4\gamma} A^{\varepsilon}_{l-1}\left(3\varepsilon^{\beta_1 - 1 + 2\beta_2} + \sum_{2 \leqq j \leqq m} \varepsilon^{2\beta_j - \beta_{j-1}}\right) \leqq A^{\varepsilon}_{l-1}\varepsilon^\gamma \tag{4.89}$$

provided ε is small enough and γ is chosen to satisfy

$$0 < 5\gamma < \min(\beta_1 + 2\beta_2 - 1, 2\beta_j - \beta_{j-1}) \tag{4.90}$$

which is possible since, by (4.65), the right hand side of (4.90) is positive.

Now the conclusion of the proof of Lemma 4.3 is obtained by (4.84) and (4.85) in the same way as above for the case $\lambda < 1$.

Next we shall go back to the proof of the Theorem itself. Take an arbitrary interval $Q \subset I$ satisfying (2.31) with some $\delta > 0$. Then by (4.52) and (4.58),

$$P_x\{x_n^\varepsilon \in Q\} = J_1^\varepsilon(\varepsilon^\gamma, n, x, Q) + \sum_{1 \leqq k \leqq n} \sum_{i_1 < \cdots < i_k} P_{i_1,\dots,i_k}^\varepsilon(\gamma, n, x, Q)$$

(4.91)

$$\leqq 2D_\delta \operatorname{mes} Q + \sum_{k=1}^{m_0} \sum_{i_1 < \cdots < i_k} P_{i_1,\dots,i_k}^\varepsilon(\gamma, n, x, Q) + \varepsilon \sum_{k=m_0+1}^{n} \binom{n}{k} \varepsilon^{(k-m_0)\gamma} \operatorname{mes} Q$$

where $m_0 =$ integral part of $(3/\gamma + 2)$, $\binom{n}{k} = n!(n-k)!/k!$ and we take

$$n = n(\varepsilon) = \text{integral part of } (\ln \varepsilon)^4. \tag{4.92}$$

If $k \leqq m_0$ and n is given by (4.92) then one of the differences $i_{l+1} - i_l$, $l = 0, \dots, k$ with $i_0 = 0$ and $i_{k+1} = n$ will be at least $(m_0 + 1)^{-1}(\ln \varepsilon)^4$ and so it will be bigger than $2(\ln \varepsilon)^2 + 1$. Hence by Corollary 4.1, if $i_{l+1} - i_l \geqq 2(\ln \varepsilon)^2 + 2$ for some $l = 0, \dots, k-1$ then

$$P_{i_1,\dots,i_k}^\varepsilon(\gamma, n, x, Q) \leqq \sup_z J_1^\varepsilon(\varepsilon^\gamma, i_{l+1} - i_l - 1, z, U_{\varepsilon^\gamma}(\tfrac{1}{2})) \leqq C_{43}\varepsilon^\gamma \tag{4.93}$$

for some $C_{43} > 0$. On the other hand, if $n - i_k \geqq 2(\ln \varepsilon)^2 + 2$ then we apply (4.52) to obtain

$$P_{i_1,\dots,i_k}^\varepsilon(\gamma, n, x, Q) \leqq 2P_{i_1,\dots,i_k}^\varepsilon(\gamma, n - N_\varepsilon, x, I) D_\delta \operatorname{mes} Q \tag{4.94}$$

where $N_\varepsilon =$ integral part of $(2(\ln \varepsilon)^2 + 2)$.

Since

$$\sum_k \sum_{i_1 < \cdots < i_k} P_{i_1,\dots,i_k}^\varepsilon(\gamma, n - N_\varepsilon, x, I) \leqq 1$$

one derives from (4.91)–(4.94) that

$$P_x\{x_n^\varepsilon \in Q\} \leqq D_\delta \operatorname{mes} Q + C_{43}\varepsilon^\gamma \sum_{k=1}^{m_0} \binom{n}{k} + \varepsilon \operatorname{mes} Q$$

(4.95)

$$\leqq 5D_\delta \operatorname{mes} Q + \varepsilon^{\gamma/2}$$

provided ε is small enough.

By the Chapman–Kolmogorov formula

$$P_x\{x_n^\varepsilon \in Q\} = \int_I \cdots \int_I P^\varepsilon(x, dy_1) P^\varepsilon(y_1, dy_2) \cdots P^\varepsilon(y_{n-2}, dy_{n-1}) P^\varepsilon(y_{n-1}, Q)$$

where $P^\varepsilon(z, \Gamma)$ is defined by (0.3). This together with (0.2) and (4.95) yield

$$\mu^\varepsilon(Q) \leqq 5D_\delta \operatorname{mes} Q + \varepsilon^{\gamma/2} \tag{4.96}$$

Since the distribution $P_x\{x_n^\varepsilon \in dy\}$ has a density and so $\mu^\varepsilon(dy)$ also has a density, then $\mu^\varepsilon(Q)=\mu^\varepsilon(\text{int}\,Q)$ where $\text{int}\,Q$ denotes the interior of Q.

The family of measures μ^ε is compact, i.e. from any sequence one can choose a weakly converging subsequence. Suppose that $\mu^{\varepsilon_i} \xrightarrow{w} \mu$, then by (4.96)

$$\mu(\text{int}\,Q) \leqq \liminf_{i\to\infty} \mu^{\varepsilon_i}(\text{int}\,Q) \leqq 5D_\delta \operatorname{mes} Q. \tag{4.97}$$

Since (4.97) holds for any closed interval Q disjoint with $\mathcal{T}_\lambda$ then it follows from (4.97) that the measure μ is absolutely continuous on $I\setminus\mathcal{T}_\lambda$ with respect to the Lebesgue measure on I. According to the Introduction (see (0.4)–(0.5)) the measure μ is f_λ-invariant.

On the other hand, according to [6] the only f_λ-invariant measure absolutely continuous with respect to Lebesgue measure on I is the measure μ_{f_λ} constructed in [6].

Therefore to prove our Theorem it suffices to show that

$$\mu(\mathcal{T}_\lambda)=0. \tag{4.98}$$

For this we shall need

Lemma 4.4. *There exist C_{44}, $\delta_1>0$ such that if $\delta \leqq \delta_1$ then*

$$P_x\{x_n^\varepsilon \in U_\delta(\mathcal{T}_\lambda)\} \leqq C_{44}(\operatorname{mes} U_\delta(\mathcal{T}_\lambda))^{1/2} \tag{4.99}$$

provided n is given by (4.92) *and ε is small enough, where*

$$U_\delta(\mathcal{T}_\lambda)=\bigcup_{z\in\mathcal{T}_\lambda} U_\delta(z).$$

Proof. Without loss of generality we can assume that

$$\delta \leqq \tfrac{1}{4}\operatorname{dist}(\tfrac{1}{2}, \mathcal{T}_\lambda) \equiv \delta_0. \tag{4.100}$$

For any Borel set Q define

$$J_\delta^\varepsilon(\delta_0,k,z,Q) \equiv P_z\{x_l^\varepsilon \notin U_{\delta_0}(\tfrac{1}{2}) \text{ for all } l=0,\dots,k-1 \text{ and } x_k^\varepsilon \in Q\}. \tag{4.101}$$

Employing Lemmas 2.6 and 2.7 in place of Lemmas 2.4 and 2.5 in the proof of Lemmas 4.1, 4.2 and Corollary 4.1, one can see for $k \geqq (\ln\varepsilon)^2$ that

$$J_\delta^\varepsilon(\delta_0,k,z,Q) \leqq C_{45} \operatorname{mes} Q \tag{4.102}$$

for some $C_{45}>0$ independent of $Q \subset U_{2\delta}(\mathcal{T}_\lambda)$, $z \in U_\delta(\mathcal{T}_\lambda)$ and δ satisfying (4.100). Notice that when x_l^ε stays all the time in $I\setminus U_{\delta_0}(\frac{1}{2})$ the situation becomes easier than in Lemma 4.1 since according to Lemma 2.1 the map f_λ acts in $I\setminus U_{\delta_0}(\frac{1}{2})$ as an expanding transformation.

Since $\operatorname{mes}\mathcal{T}_\lambda=0$ (see [6] Lemma 3.9) and $\mathcal{T}_\lambda$ is a closed set then $\operatorname{mes}(U_\delta(\mathcal{T}_\lambda)) \downarrow 0$ as $\delta \downarrow 0$, and so if δ is small enough then $\operatorname{mes}(U_\delta(\mathcal{T}_\lambda)) \leqq 1/2C_{45}$.

Thus by (4.102),

$$P_z\{x_l^\varepsilon \notin U_{\delta_0}(\tfrac{1}{2}) \text{ for all } l=0,\dots,k-1 \text{ and } x_k^\varepsilon \in U_\delta(\mathcal{T}_\lambda)\} = J_6^\varepsilon(\delta_0, k, z, U_\delta(\mathcal{T}_\lambda)) \leqq C_{45} \operatorname{mes} U_\delta(\mathcal{T}_\lambda) \leqq \tfrac{1}{2} \tag{4.103}$$

provided $k \geqq (\ln \varepsilon)^2$ and ε is small enough.

Modifying slightly arguments of Lemma 4.3 one can see that

$$P_x\{x_m^\varepsilon \in U_\delta(\mathcal{T}_\lambda)\} \leqq P_x\{x_l^\varepsilon \in U_{\varepsilon^\gamma}(\tfrac{1}{2}) \text{ for all } l \geqq \tfrac{1}{3}(\ln \varepsilon)^4 \text{ and } x_m^\varepsilon \in U_\delta(\mathcal{T}_\lambda)\} + \varepsilon^{\gamma/2}. \tag{4.104}$$

By Corollary 4.1,

$$P_x\{x_l^\varepsilon \notin U_{\varepsilon^\gamma}(\tfrac{1}{2}) \text{ for all } l \geqq \tfrac{1}{3}(\ln \varepsilon)^4 \text{ and } x_k^\varepsilon \in Q\} \leqq 2D_\delta \operatorname{mes} Q \tag{4.105}$$

provided $k \geqq \frac{2}{3}(\ln \varepsilon)^4$ and Q satisfies (2.31). Besides, by (4.103)

$$P_z\{x_l^\varepsilon \notin U_{\delta_0}(\tfrac{1}{2}) \text{ for all } l=0,\dots,k\} \leqq (\tfrac{1}{2})^{(\ln \varepsilon)^2/4} \qquad \text{if } k \geqq \tfrac{1}{4}(\ln \varepsilon)^4. \tag{4.106}$$

Next, by (4.104)–(4.106) and (4.52) one can write

$$\begin{aligned} &P_x\{x_n^\varepsilon \in U_\delta(\mathcal{T}_\lambda)\} \\ &\leqq \sum_{1/3(\ln \varepsilon)^4 \geqq k \geqq 1} \int_{U_{\delta_0}(1/2)\setminus U_{\varepsilon^{1/4}}(1/2)} P_x\{x_l^\varepsilon \notin U_{\varepsilon^\gamma}(\tfrac{1}{2}) \text{ if } n-k \geqq l \geqq \tfrac{1}{3}(\ln \varepsilon)^4 \text{ and } x_{n-k}^\varepsilon \in dy\} \\ &\quad \times \int_{I\setminus U_{\delta_0}(1/2)} q_{f_\lambda y}^\varepsilon(z) J_6^\varepsilon(\delta_0, k-1, z, U_\delta(\mathcal{T}_\lambda))dz + C_{47} n\varepsilon^{1/4} + n\varepsilon^\gamma + n(\tfrac{1}{2})^{(\ln \varepsilon)^4/4} \\ &\leqq n\varepsilon^\gamma + n(\tfrac{1}{2})^{(\ln \varepsilon)^4/4} + C_{46} n\varepsilon^{1/4} \\ &\quad + C_{47} \sum_{1/3(\ln \varepsilon)^4 \geqq k \geqq 1} \int_{U_{\delta_0}(1/2)\setminus U_{\varepsilon^{1/4}}(1/2)} q_{f_\lambda y}^\varepsilon(z) \int_{I\setminus U_{\delta_0}(1/2)} J_6^\varepsilon(\delta_0, k-1, z, U_\delta(\mathcal{T}_\lambda))dy\,dz \end{aligned} \tag{4.107}$$

where n is defined by (4.92) and $C_{46}, C_{47} > 0$ are independent of x, y, n and δ. Employing (4.70) and changing variables $v = f_\lambda y$ one obtains

$$\begin{aligned} &\int_{U_{\delta_0}(1/2)\setminus U_{\varepsilon^{1/4}}(1/2)} q_{f_\lambda y}^\varepsilon(z) \int_{I\setminus U_{\delta_0}(1/2)} J_6(\delta_0, k-1, z, U_\delta(\mathcal{T}_\lambda))dy\,dz \\ &\leqq C_{48} \int_{U_{5\lambda\delta_0^2}(f_\lambda(1/2))\setminus U_{3\lambda\varepsilon^{1/2}}(f_\lambda(1/2))} q_v^\varepsilon(z)|f_\lambda(\tfrac{1}{2}) - v|^{-1/2} dv \int_{I\setminus U_{\delta_0}(1/2)} J_6^\varepsilon(\delta_0, k-1, z, U_\delta(\mathcal{T}_\lambda))dz \\ &\leqq C_{49} \int_{U_{6\lambda\delta_0^2}(f_\lambda(1/2))\setminus U_{2\lambda\varepsilon^{1/2}}(f_\lambda(1/2))} |f_\lambda(\tfrac{1}{2}) - z|^{-1/2} J_6^\varepsilon(\delta_0, k-1, z, U_\delta(\mathcal{T}_\lambda))dz \\ &\equiv C_{49} J_7^\varepsilon(\delta_0, k-1, U_\delta(\mathcal{T}_\lambda)) \end{aligned} \tag{4.108}$$

where we could integrate $q_v^\varepsilon(z)$ in v without $|f_\lambda(\frac{1}{2})-v|^{-1/2}$ since $q_v^\varepsilon(z)$ varies essentially only when $v \in U_{\varepsilon^{1-\beta}}(z)$ where $\beta > 0$ can be taken small, and if, in addition, $v \notin U_{3\lambda\varepsilon^{1/2}}(f_\lambda(\frac{1}{2}))$ then the ratio $|f_\lambda(\frac{1}{2})-v|/|f_\lambda(\frac{1}{2})-z|$ is close to one.

Let $h_1^\varepsilon(\delta_0, l, z, v)$ be the density of $J_6^\varepsilon(\delta_0, l, z, Q)$, i.e.

$$J_6^\varepsilon(\delta_0, l, z, Q) = \int_Q h_1^\varepsilon(\delta_0, l, z, v)dv.$$

Define $m(z) = \min\{l: |f_\lambda(\frac{1}{2})-z|\,|(f_\lambda^{l+1})'(z)| \geqq \delta_0\}$. Then if $k-1 > m(z)$ one has

$$J_6^\varepsilon(\delta_0, k-1, z, U_\delta(\mathcal{T}_\lambda))$$

$$= \int_{I\setminus U_{\delta_0}(1/2)} dv\, h_1^\varepsilon(\delta_0, m(z), z, v) \int_{U_\delta(\mathcal{T}_\lambda)} h_1^\varepsilon(\delta_0, k-m(z)-1, v, w)dw.$$

(4.109)

Employing the argument about pseudo-orbits similar to the inequality (4.5), which says that it suffices to take into account only random trajectories staying in the $\varepsilon^{1-\beta}$-neighborhood of images $f_\lambda^i y$ for certain points y, one concludes that

$$h_1^\varepsilon(\delta_0, m(z), z, v) \leqq h_2^\varepsilon(\varepsilon^{1-\beta}, m(z); z, v; y) + \exp(-\varepsilon^{-\alpha_9\beta}) \tag{4.110}$$

for some $\alpha_9 > 0$ where $y = U_{\varepsilon^{1-\beta}}(z) \cap f_\lambda^{-m(z)}v$ and

$$h_2^\varepsilon(\varepsilon^{1-\beta}, l; z, v; y) = \int_{U_{\varepsilon^{1-\beta}}(f_\lambda y)} \cdots \int_{U_{\varepsilon^{1-\beta}}(f_\lambda^{-l} y)} q_{f_\lambda z}(z_1)\cdots q_{f_\lambda z_{l-1}}(v)dz_1 \cdots dz_{l-1}.$$

Using Lemma 3.3 in the same way as in the proof of Lemma 4.1 one can see that

$$h_2^\varepsilon(\varepsilon^{1-\beta}, l; z, v; y) \leqq C_{50}\varepsilon^{-1}|(f_\lambda^l)'(y)|^{-1}\exp\{-\alpha_4\varepsilon^{-1}|z-y|\} \tag{4.111}$$

for some $C_{50} > 0$ independent of ε, l, z, v and y.

Now taking into account (4.110), (4.111), the argument (as above) that $\varepsilon^{-1}\exp\{-\alpha_4\varepsilon^{-1}|z-y|\}$ can be integrated in z without $|f_\lambda(\frac{1}{2})-z|^{-1/2}$ and that the difference $|m(z)-m(y)|$ is bounded when $z \notin U_{2\lambda\varepsilon^{1/2}}(f_\lambda(\frac{1}{2}))$ and $|z-y| \leqq \varepsilon^{3/4}$, we obtain

$$h_3^\varepsilon(\delta_0, l, v) \equiv \int_{U_{6\lambda\delta_0^2}(f_\lambda(1/2))\setminus U_{2\lambda\varepsilon^{1/2}}(f_\lambda(1/2))} |f_\lambda(\tfrac{1}{2})-z|^{-1/2}h_1^\varepsilon(\delta_0, l, z, v)dz$$

$$\leqq C_{51}|f_\lambda(\tfrac{1}{2})-y(v)|^{-1/2}|(f_\lambda^l)'(y(v))|^{-1} \tag{4.112}$$

$$\leqq C_{52}\gamma_2^{-l}|f_\lambda^{l+1}(\tfrac{1}{2})-v|^{-1/2}$$

where $y(v) = U_{\varepsilon^{1-\beta}}(z) \cap f_\lambda^{-l}v$; $C_{51}, C_{52} > 0$ and $\gamma_2 > 1$.

If $l = m(y(v))$ then one can see from (4.112), Lemma 2.6 and the definition of $m(y(v))$ that

$$h_3^\varepsilon(\delta_0, l, v) \leqq C_{53}\gamma_2^{-l} \tag{4.113}$$

where $C_{53} > 0$ depends only on δ_0. Hence

$$J_7^\varepsilon(\delta_0, k-1, U_\delta(\mathcal{T}_\lambda))$$

$$\leqq C_{53} \int_{\{v\in I\setminus U_{\delta_0}(1/2), m(y(v))<k-1\}} \gamma_2^{-m(y(v))} dv \int_{U_\delta(\mathcal{T}_\lambda)} h_1^\varepsilon(\delta_0, k-m(y(v))-1, v, w)dw \tag{4.114}$$

$$+ C_{52}\gamma_2^{-(k-1)} \int_{U_\delta(\mathcal{T}_\lambda)} |f_\lambda^k(\tfrac{1}{2}) - v|^{-1/2} dv + \exp(-\varepsilon^{-\alpha_9\beta}).$$

But

$$\int_{U_\delta(\mathcal{T}_\lambda)} |f_\lambda^k(\tfrac{1}{2}) - v|^{-1/2} dv$$

$$\leqq \int_{\{v: |f_\lambda^k(1/2)-v|\leqq \text{mes}\, U_\delta(\mathcal{T}_\lambda)\}} |f_\lambda^k(\tfrac{1}{2}) - v|^{-1/2} dv + \int_{\{v: |f_\lambda^k(1/2)-v|\geqq \text{mes}\, U_\delta(\mathcal{T}_\lambda), v\in U_\delta(\mathcal{T}_\lambda)\}} |f_\lambda^k(\tfrac{1}{2}) - v|^{-1/2} dv$$

$$\leqq 3(\text{mes}\, U_\delta(\mathcal{T}_\lambda))^{1/2}. \tag{4.115}$$

To estimate the first term in (4.114) we shall use, as in (4.110), the argument about pseudo-orbits to obtain that

$$h_1^\varepsilon(\delta_0, k-m(y(v))-1, v, w) \leqq \sum_{y_i} h_2^\varepsilon(\varepsilon^{1-\beta}, k-m(y(v))-1, v, w; y_i) + \exp(-\varepsilon^{-\alpha_9\beta}), \tag{4.116}$$

where $\{y_i\} = U_{\varepsilon^{1-\beta}}(v) \cap f_\lambda^{-(k-m(z)-1)}w$ such that $f^j y_i \notin U_{\delta_0}(\frac{1}{2})$ for all $j = 0, \ldots, k-m(z)-1$.

Notice that $|m(y(v)) - m(y(y_i))| \leqq C_{54}$ for some $C_{54} > 0$ independent of v and i. Then by (4.111),

$$\int_{\{v\in I\setminus U_{\delta_0}(1/2), m(y(v))<k-1\}} \gamma_2^{-m(y(v))} h_1^\varepsilon(\delta_0, k-m(y(v))-1, v, w)dv$$

$$\leqq C_{55}\gamma_2^{-m(y(y_1))} \sum_i |(f_\lambda^{k-m(y(y_1))-1})'(y_i)|^{-1} + \exp(-\varepsilon^{-\alpha_9\beta}) \tag{4.117}$$

for some $C_{55} > 0$.

It is easy to see that

$$\sum_i |(f_\lambda^{k-m(y(y_1))-1})'(y_i)|^{-1} \leqq \operatorname{mes}\{x \in I : f_\lambda^l x \in U_{\delta_0/2}(\tfrac{1}{2}) \text{ for all } l = 0, \ldots, k - m(y(y_1)) - 1\}$$

$$\leqq \gamma_3^{-(k-m(y(y_1))-1} \tag{4.118}$$

for some $\gamma_3 > 1$ where the last inequality follows from Proposition 2.1 of [6].

Collecting (4.115), (4.116), (4.118) and (4.119) one obtains

$$J_7^\varepsilon(\delta_0, k-1, U_\delta(\mathcal{T}_\lambda)) \leqq C_{56}\gamma_4^{-(k-1)}((\operatorname{mes} U_\delta(\mathcal{T}_\lambda))^{1/2} + \operatorname{mes} U_\delta(\mathcal{T}_\lambda))$$

where $\gamma_4 = \min(\gamma_2, \gamma_3)$ and $C_{56} > 0$; this, together with (4.107) and (4.108), proves (4.99) provided ε is small enough.

To complete the proof of (4.98) we employ (0.2), the Chapman–Kolmogorov formula (4.96) and (4.99) to obtain

$$\mu^\varepsilon(U_\delta(\mathcal{T}_\lambda)) \leqq C_{51}(\operatorname{mes} U_\delta(\mathcal{T}_\lambda))^{1/2}. \tag{4.120}$$

If $\mu^{\varepsilon_i} \xrightarrow{w} \mu$ then by (4.120),

$$\mu(\mathcal{T}_\lambda) \leqq \mu(\operatorname{int} U_\delta(\mathcal{T}_\lambda)) \leqq \liminf_{i\to\infty} \mu^{\varepsilon_i}(\operatorname{int} U_\delta(\mathcal{T}_\lambda)) \leqq C_{51}(\operatorname{mes} U_\delta(\mathcal{T}_\lambda))^{1/2}. \tag{4.121}$$

Since $\operatorname{mes} \mathcal{T}_\lambda = 0$ (see Lemma 3.9 of [6]) then $\operatorname{mes} U_\delta(\mathcal{T}_\lambda) \to 0$ as $\delta \to 0$ and, together with (4.121), this gives (4.98).

Appendix

We shall prove here the following

Proposition A.1. *Let $\theta_1, \ldots, \theta_n$ be independent random variables with distribution functions*

$$P\{\theta_k \leqq y\} = \int_{-\infty}^{y} r_k(z)dz,$$

where the number of points of discontinuity of each $r_k(z)$ is bounded by a number N independent of k, on each interval of continuity $r_k(z)$ are Lipschitz continuous with a constant L and

$$r_k(z) \leqq Ce^{-\alpha|z|} \tag{A.1}$$

with some C, $\alpha > 0$ independent of k. Then there exist constants K, $\beta > 0$ depending only on L, N, C, α but independent of n such that for any non-zero numbers $a_1, \ldots, a_n$ the distribution function of the random variable

$$\Psi^{a_1,\ldots,a_n} = \Big(\sum_{1\leqq k\leqq n} a_k^2\Big)^{-1/2} \sum_{1\leqq k\leqq n} a_k(\theta_k - E\theta_k) \tag{A.2}$$

has the derivative, i.e. the probability density function, satisfying

$$r^{a_1,\dots,a_n}(z) \leqq Ke^{-\beta|z|} \tag{A.3}$$

where $E\theta_k = \int_{-\infty}^{\infty} zr_k(z)dz$ is the expectation of θ_k.

Proof. Without loss of generality we shall assume from the beginning that $E\theta_k = 0$ for all k. We shall first prove the integral variant of the inequality (A.3) and then, employing the Fourier transform, we shall see that $r^{a_1,\dots,a_n}$ has a bounded derivative for $n \geqq 3$ which will imply Proposition A.1.

Lemma A.1 (cf. [7], §4, ch. III). *There exist K_1, $\beta_1 > 0$ such that*

$$P\{|\Psi^{a_1,\dots,a_n}| > y\} \leqq K_1 e^{-\beta_1 y} \tag{A.4}$$

for any $y \geqq 0$.

Proof. By (A.1),

$$E|\theta_k^m| \leqq \int_{-\infty}^{\infty} C|z|^m e^{-\alpha|z|}dz = 2C\int_0^{\infty} z^m e^{-\alpha z}dz = \frac{2C}{\alpha^{m+1}}\, m!. \tag{A.5}$$

Put

$$\sigma_k = \Big(\sum_{1\leqq k\leqq n} a_k^2\Big)^{-1/2} |a_k| \quad \text{and} \quad X_k = \sigma_k\theta_k, \tag{A.6}$$

then by (A.5),

$$\begin{aligned} Ee^{tX_k} &\leqq 1 + \frac{t^2}{2!}EX_k^2 + \frac{|t|^3}{3!}E|X_k|^3 + \cdots \\ &\leqq 1 + 2C\alpha^{-1}\sum_{m=2}^{\infty} |t|^m\sigma_k^m\alpha^{-m} \\ &= 1 + \frac{2Ct^2\sigma_k^2\alpha^{-3}}{1-|t|\sigma_k\alpha^{-1}} \end{aligned}$$

provided $|t|$ is small enough. If $|t| \leqq \alpha/2$ then $1-|t|\sigma_k\alpha^{-1} \geqq \frac{1}{2}$ and we can write

$$Ee^{tX_k} \leqq 1 + 4Ct^2\sigma_k^2\alpha^{-3} < e^{4Ct^2\sigma_k^2\alpha^{-3}} \tag{A.7}$$

Thus for t, $y \geqq 0$,

$$P\{\Psi^{a_1,\dots,a_n} \geqq y\} = P\{e^{t\Psi^{a_1,\dots,a_n}} \geqq e^{ty}\} \leqq e^{-ty}\prod_{k=1}^{n} Ee^{tX_k} \leqq e^{-ty}e^{4Ct^2\alpha^{-3}} \tag{A.8}$$

where we have used the independency of X_k, $k = 1,\dots,n$ and the relation $\sum_{k=1}^{n}\sigma_k^2 = 1$. Furthermore for $t > 0$,

$$P\{\Psi^{a_1,\dots,a_n} \leqq -y\} = P\{(-t\Psi^{a_1,\dots,a_n}) \geqq ty\}$$

$$= P\{e^{-t\Psi^{a_1,\dots,a_n}} \geqq e^{ty}\} \tag{A.9}$$

$$\leqq e^{-ty}\prod_{k=1}^{n} Ee^{(-t)X_k} \leqq e^{-ty}e^{4Ct^2\alpha^{-3}}.$$

Taking $t = \alpha/2$ in (A.8) and (A.9) we shall get (A.4) with $\beta_1 = \alpha/2$ and $K_1 = e^{C\alpha^{-1}}$, proving Lemma A.1.

Next, we shall need certain estimates of the Fourier transform

$$\varphi_k(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta z} r_k(z)dz$$

of densities r_i for real ζ. We shall start with the estimate near zero which is simpler. By (A.1) one has the following estimate for the m-th derivative of φ_i at zero,

$$\varphi_k^{(m)}(0) \leqq \int_{-\infty}^{\infty} |z|^m r_k(z)dz \leqq \frac{2Cm!}{\alpha^{m+1}}.$$

Since we suppose that

$$E\theta_k = \int_{-\infty}^{\infty} zr_k(z)dz = 0 \tag{A.10}$$

then for $|\zeta| < \alpha$ it follows that

$$\left|\varphi_k(\zeta) - 1 + \frac{\zeta^2}{2}\right| \leqq 2C\alpha^{-1}\sum_{m=3}^{\infty}\left(\frac{|\zeta|}{\alpha}\right)^m = |\zeta|^2\left(\frac{2C|\zeta|}{\alpha^3(\alpha - |\alpha|)}\right). \tag{A.11}$$

In particular, for a real ζ satisfying

$$|\zeta| < \frac{\alpha^4}{8C + \alpha^3} \tag{A.12}$$

we have

$$|\varphi_k(\zeta)| \leqq 1 - \zeta^2/4. \tag{A.13}$$

Estimates for big ζ are a little bit more difficult. We shall prove the following result.

Lemma A.2. *There exists a constant $K_2 > 0$ depending only on L, N, C and α such that*

$$|\varphi_k(\zeta)| \leqq \frac{K_2 \log|\zeta|}{|\zeta|} \tag{A.14}$$

provided $|\zeta| > 2$.

Proof. We can write

$$\varphi_k(\zeta) = \int_{-\infty}^{-\alpha^{-1}\log|\zeta|} e^{i\zeta z} r_k(z)dz + \int_{\alpha^{-1}\log|\zeta|}^{\infty} e^{i\zeta z} r_k(z)dz + \int_{-\alpha^{-1}\log|\zeta|}^{\alpha^{-1}\log|\zeta|} e^{i\zeta z} r_k(z)dz. \tag{A.15}$$

In view of (A.1) we can estimate the first term in (A.15) by

$$C \int_{-\infty}^{-\alpha^{-1}\log|\zeta|} e^{-\alpha|z|}dz = C/|\zeta|. \tag{A.16}$$

The second term in (A.15) has the same estimate.

To estimate the third term in (A.15) consider the points of discontinuity for the function r_k lying between $-\alpha^{-1}\log|\zeta|$ and $\alpha^{-1}\log|\zeta|$. Number them in increasing order $z_1, \dots, z_l$; by assumption $l \leqq N$. Thus we can write

$$\left| \int_{-\alpha^{-1}\log|\zeta|}^{\alpha^{-1}\log|\zeta|} e^{i\zeta z} r_k(z)dz \right| \leqq \sum_{j=0}^{l} \left| \int_{z_j}^{z_{j+1}} e^{i\zeta z} r_k(z)dz \right| \tag{A.17}$$

where we put $z_0 = -\alpha^{-1}\log|\zeta|$ and $z_{l+1} = \alpha^{-1}\log|\zeta|$. Divide the interval $[z_j, z_{j+1}]$ into subintervals of the same length $2\pi/|\zeta|$, except for the last interval whose length does not exceed $2\pi/|\zeta|$. Let Δ be one of these intervals of length $2\pi/|\zeta|$, then by the Lipschitz condition

$$\left| \int_\Delta e^{i\zeta z} r_k(z)dz \right| \leqq \int_\Delta |r_k(z) - r_k^\Delta|\,dz + r_k^\Delta \left| \int_\Delta e^{i\alpha z}dz \right| \leqq 4\pi^2 L/|\zeta|^2 \tag{A.18}$$

where $r_k^\Delta = \int_\Delta r_k(z)dz$ and we take into account that $\int_\Delta e^{i\zeta z}dz = 0$ since the length of Δ equals $2\pi/|\zeta|$. The total number of these subintervals in $[z_0, z_{l+1}]$ equals the integral part of $(1/2\pi)(z_{j+1} - z_j)\zeta$ so that the sum of our estimates in (A.18) does not exceed $(2\pi L/|\zeta|)(z_{j+1} - z_j)$. The contribution of the remaining subinterval with the length less than or equal to $2\pi/|\zeta|$ does not exceed $2\pi C/|\zeta|$. Therefore

$$\left| \int_{z_j}^{z_{j+1}} e^{i\zeta z} r_k(z)dz \right| \leqq \frac{2\pi}{|\zeta|}(C + L(z_{j+1} - z_j)). \tag{A.19}$$

By (A.17) and (A.19) we can estimate the third term in (A.15) by

$$\frac{2\pi}{|\zeta|}(NC + 2L\alpha^{-1}\log|\zeta|) = \frac{\log|\zeta|}{|\zeta|}\left(\frac{2\pi NC}{\log|\zeta|} + 2L\alpha^{-1}\right).$$

Since the first and second terms of (A.15) are bounded by $C/|\zeta|$ we shall get (A.14) with

$$K_2 = \frac{2\pi(N+2)C}{\log 2} + 2L\alpha^{-1},$$

completing the proof of Lemma A.2.

Next, we shall need an estimate for $\varphi_k(\zeta)$ for the intermediate zone, i.e. for ζ not very big and not very small.

Lemma A.3 (see [7], Theorem 1 in ch. I). *Let $\varphi(\zeta)$ be the Fourier transform of a probability distribution G, i.e. $\varphi(\zeta)=\int_{-\infty}^{\infty} e^{i\zeta z}dG(z)$, $c<1$ and b are positive constants. If $|\varphi(\zeta)|\leqq c$ for $|\zeta|\geqq b$ then*

$$|\varphi(\zeta)|\leqq 1-\frac{1-c^2}{8b^2}\zeta^2 \qquad \text{for } |\zeta|<b. \tag{A.20}$$

Proof. Let X and $\tilde{X}$ be two independent random variables with the distribution G. Then, as is well known, the random variable $X-\tilde{X}$ has the probability distribution Q whose Fourier transform equals $|\varphi(\zeta)|^2$. Then we have

$$1-|\varphi(\zeta)|^2=\int_{-\infty}^{\infty}(1-\cos\zeta z)dQ(z).$$

Clearly,

$$1-\cos\zeta z=2\sin^2\frac{\zeta z}{2}\geqq\tfrac{1}{4}(1-\cos 2\zeta z)$$

and so

$$1-|\varphi(2\zeta)|^2\leqq 4(1-|\varphi(\zeta)|^2)$$

which implies

$$1-|\varphi(2^m\zeta)|^2\leqq 4^m(1-|\varphi(\zeta)|^2) \tag{A.21}$$

for any positive integer m.

For $\zeta=0$ the inequality (A.20) is clear. Let now $0<|\zeta|<b$. Choose n so that $2^{-m}b\leqq|\zeta|<2^{-m+1}b$. Then by Assumption $|\varphi(2^m\zeta)|^2\leqq c^2$ and so by (A.21) it follows that

$$|\varphi(\zeta)|^2\leqq 1-\frac{1-c^2}{4^n}\leqq 1-\frac{1-c^2}{4b^2}\zeta^2.$$

Hence

$$|\varphi(\zeta)|\leqq 1-\frac{1-c^2}{8b^2}\zeta^2$$

proving Lemma A.3

Now we come back to the proof of Proposition A.1. We are going to estimate the Fourier transform $\varphi^{a_1,\dots,a_n}(\zeta)$ of the probability density $r^{a_1,\dots,a_n}$ of the random variable $\Psi^{a_1,\dots,a_n}$ defined by (A.2). Since the random variables $\theta_1,\dots,\theta_n$ are independent then

$$\varphi^{a_1,\dots,a_n}(\zeta)=\prod_{k=1}^{n}\varphi_k(\sigma_k\zeta) \tag{A.22}$$

where σ_k is defined by (A.6).

Let

$$\mathcal{K}_1(\zeta)\equiv\left\{k:\ \sigma_k|\zeta|<\frac{\alpha^4}{8C+\alpha^3}\right\},$$

$$\mathcal{K}_2(\zeta)\equiv\left\{k:\ \frac{\alpha^4}{8C+\alpha^3}\geqq\sigma_k|\zeta|<\max\left(2,\frac{|\zeta|}{\log|\zeta|}\right)\right\}\qquad\text{and}$$

$$\mathcal{K}_3(\zeta)\equiv\left\{k:\ \sigma_k|\zeta|\geqq\max\left(2,\frac{|\zeta|}{\log|\zeta|}\right)\right\}$$

where we put $\mathcal{K}_2(\zeta)=\varnothing$ if $\alpha^4/(8C+\alpha^3)\geqq\max(2,|\zeta|/\log|\zeta|)$. Since $\Sigma_{k=1}^n\sigma_k^2=1$ then at least one of the following three inequalities is true:

$$\sum_{k\in\mathcal{K}_1(\zeta)}\sigma_k^2\geqq\tfrac{1}{3},\qquad\sum_{k\in\mathcal{K}_2(\zeta)}\sigma_k^2\geqq\tfrac{1}{3}\quad\text{or}\quad\sum_{k\in\mathcal{K}_3(\zeta)}\sigma_k^2\geqq\tfrac{1}{3}. \tag{A.23}$$

We shall consider three cases. Suppose that the first inequality in (A.23) is satisfied. Employing (A.13) for $k\in\mathcal{K}_1(\zeta)$ we have from (A.22) that

$$\begin{aligned}|\varphi^{a_1,\dots,a_n}(\zeta)|&\leqq\prod_{k\in\mathcal{K}_1(\zeta)}|\varphi_k(\sigma_k\zeta)|\\&\leqq\prod_{k\in\mathcal{K}_1(\zeta)}\left(1-\frac{\sigma_k^2\zeta^2}{4}\right)\leqq\prod_{k\in\mathcal{K}_1(\alpha)}e^{-\sigma_k^2\zeta^2/4}\leqq e^{-\zeta^2/12}\end{aligned} \tag{A.24}$$

since $1-a\leqq e^{-a}$.

To estimate $\varphi^{a_1,\dots,a_n}$ in the second case notice, first, that one can choose $K_3>1$ such that

$$K_2|\zeta|^{-1/2}\log|\zeta|\leqq 1\qquad\text{if }|\zeta|\geqq K_3$$

and so by (A.14),

$$\varphi_k(\zeta)\leqq|\zeta|^{-1/2}\qquad\text{if }|\zeta|\geqq K_3. \tag{A.25}$$

Applying (A.20) when $K_3>|\zeta|\geqq\alpha^4/(8C+\alpha^3)$ and (A.25) when $|\zeta|\geqq K_3$ one concludes that

$$|\varphi_k(\zeta)|\leqq\max\left(K_3^{-1/2},1-\frac{\alpha^8(1-K_3^{-1})}{8K_3^2(8C+\alpha^3)^2}\right)\equiv\gamma<1 \tag{A.26}$$

provided $|\zeta|\geqq\alpha^4/(8C+\alpha^3)$. Hence

$$|\varphi_k(\sigma_k\zeta)|\leqq\gamma\qquad\text{if }k\in\mathcal{K}_2(\zeta). \tag{A.27}$$

Suppose that the second inequality in (A.23) holds true. Since

$$\sigma_k < \max(2|\zeta|^{-1}, (\log|\zeta|)^{-1}) \qquad \text{if } k \in \mathcal{K}_2(\zeta),$$

then the number $\nu_2(\zeta)$ of elements in $\mathcal{K}_2(\zeta)$ satisfies

$$\frac{1}{3} \leqq \sum_{k \in \mathcal{K}_2(\zeta)} \sigma_k^2 < \nu_2(\zeta) \max(4|\zeta|^{-2}, (\log|\zeta|)^{-2}),$$

i.e.

$$\nu_2(\zeta) > \tfrac{1}{3} \min(\tfrac{1}{4}|\zeta|^2, (\log|\zeta|)^2). \tag{A.28}$$

By (A.22) and (A.27) we see that

$$|\varphi^{a_1,\dots,a_n}(\zeta)| \leqq \prod_{k \in \mathcal{K}_2(\zeta)} |\varphi_k(\sigma_k \zeta)| \leqq \gamma^{\nu_2(\zeta)}. \tag{A.29}$$

From (A.28) it follows that for each $l > 0$ there exists $K_4(l) > 0$ such that $\gamma^{\nu_2(\zeta)} \leqq K_4(l)|\zeta|^{-l}$. In particular, one can write

$$|\varphi^{a_1,\dots,a_n}(\zeta)| \leqq K_4(3)|\zeta|^{-3}. \tag{A.30}$$

Finally, we shall consider the case when the third inequality in (A.23) holds true. When $k \in \mathcal{K}_3(\zeta)$ we can estimate $\varphi_k(\sigma_k \zeta)$ by means of (A.14) to obtain

$$\begin{aligned} |\varphi^{a_1,\dots,a_n}(\zeta)| &\leqq \prod_{k \in \mathcal{K}_3(\zeta)} |\varphi_k(\sigma_k \zeta)| \\ &\leqq K_2^{\nu_3(\zeta)} |\zeta|^{-\nu_3(\zeta)} \prod_{k \in \mathcal{K}_3(\zeta)} \sigma_k^{-1} \log(\sigma_k |\zeta|) \end{aligned} \tag{A.31}$$

where $\nu_3(\zeta)$ is the number of elements in $\mathcal{K}_3(\zeta)$. Since $1 > \sigma_k \geqq (\log|\zeta|)^{-1}$ if $k \in \mathcal{K}_3(\zeta)$ and $\log 2 \leqq \log(\sigma_k|\zeta|) \leqq \log|\zeta|$, then we derive from (A.31) that

$$|\varphi^{a_1,\dots,a_n}(\zeta)| \leqq K_2^{\nu_3(\zeta)} |\zeta|^{-\nu_3(\zeta)} (\log|\zeta|)^{2\nu_3(\zeta)}. \tag{A.32}$$

There exists $K_5 > 0$ such that

$$\sup_{|\zeta| \geqq K_5} (K_2|\zeta|^{-1/4}(\log|\zeta|)^2) = \tilde{\gamma} < 1.$$

Then (A.32) implies

$$|\varphi^{a_1,\dots,a_n}(\zeta)| \leqq \begin{cases} 1 & \text{if } |\zeta| < K_5, \\ |\zeta|^{-3\nu_3(\zeta)/4} & \text{if } |\zeta| \geqq K_5. \end{cases} \tag{A.33}$$

In particular, if $\nu_3(\zeta) \geqq 3$ we get the estimate

$$|\varphi^{a_1,\dots,a_n}(\zeta)| \leqq |\zeta|^{-9/4} \qquad \text{if } |\zeta| \geqq K_5. \tag{A.34}$$

The case $\nu_3(\zeta) = 1$ or 2 must be treated separately. In this case there exists $k_0 \in \mathcal{K}_3(\zeta)$ such that $\sigma_{k_0} \geqq 1/\sqrt{6}$. For convenience of notation only assume that

$k_0 = n$. Then we can write, using (A.13),

$$r^{a_1,\dots,a_n}(z) = \int_{-\infty}^{\infty} P\left\{\sum_{k=1}^{n-1} \sigma_k\theta_k \in dy\right\} \frac{1}{\sigma_n} r_n\left(\frac{z-y}{\sigma_n}\right)$$

$$\text{(A.35)} \qquad = \int_{\{y:|z-y|<|z|/2\}} + \int_{\{y:|z-y|\geqq|z|/2\}}$$

$$\leqq K\sqrt{6}\,P\left\{\left|\sum_{k=1}^{n-1} \sigma_k\theta_k\right| > |z|/2\right\} + K\sqrt{6}\,e^{-\beta|z|/2}.$$

Using the notation $\Psi^{\sigma_1,\dots,\sigma_{n-1}}$ defined by (A.2) we have

$$\sum_{k=1}^{n-1} \sigma_k\theta_k = \left(\sum_{k=1}^{n-1} \sigma_k^2\right)^{1/2} \Psi^{\sigma_1,\dots,\sigma_{n-1}}.$$

Since $\Sigma_{k=1}^{n-1} \sigma_k^2 \leqq 1$, then employing Lemma A.1 we conclude

$$P\left\{\left|\sum_{k=1}^{n-1} \sigma_k\theta_k\right| > |z|/2\right\} \leqq P\{|\Psi^{\sigma_1,\dots,\sigma_{n-1}}| > |z|/2\}$$

$$\text{(A.36)} \qquad \leqq K_1 e^{-\beta_1|z|/2}.$$

Now (A.35) and (A.36) give (A.3) directly for the case when there exists k_0 such that $\sigma_{k_0} \geqq 1/\sqrt{6}$. If this is not true then $\varphi^{a_1,\dots,a_n}$ satisfies one of the inequalities (A.24), (A.30) or (A.34). These inequalities show then that there exists a constant $K_6 > 0$ independent of $a_1, \dots, a_n$ and n such that

$$\text{(A.37)} \qquad \int_{-\infty}^{\infty} |\zeta|\,|\varphi^{a_1,\dots,a_n}(\zeta)|\,d\zeta \leqq K_6 < \infty.$$

Using the inverse Fourier transform formula we see that

$$r^{a_1,\dots,a_n}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\zeta z} \varphi^{a_1,\dots,a_n}(\zeta)\,d\zeta$$

and so, by (A.37),

$$\text{(A.38)} \qquad \left|\frac{d}{dz} r^{a_1,\dots,a_n}(z)\right| \leqq \frac{1}{2\pi} K_6.$$

From (A.4) it follows for $z \leqq 0$ that

$$K_1 e^{-\beta_1 z} \geqq P\{z < \Psi^{a_1,\dots,a_n} \leqq z + e^{-\beta_1 z/2}\}$$

$$\text{(A.39)} \qquad = \int_z^{z+e^{-\beta_1 z/2}} r^{a_1,\dots,a_n}(y)\,dy$$

$$\geqq r^{a_1,\dots,a_n}(z)\,e^{-\beta_1 z/2} - \frac{K_6}{2\pi}\,e^{-\beta_1 z}$$

since, by (A.38),

$$|r(y)-r(z)| \leqq \frac{K_6}{2\pi} e^{-\beta_1 z/2} \qquad \text{when } y \in [z, z+e^{-\beta_1 z/2}].$$

Therefore

$$r^{a_1,\dots,a_n}(z) \leqq \left(K_1 + \frac{K_6}{2\pi}\right) e^{-\beta_1 z/2}$$

completing the proof of (A.3) for $z \geqq 0$. For negative z the proof remains the same by considering the integral from $z - e^{-\beta_1 z/2}$ to z.

References

1. A. Boyarsky, *Randomness implies order*, J. Math. Anal. Appl. **76** (1980), 483–497.
2. P. Collet, *Ergodic properties of some unimodal mappings of the interval*, Institut Mittag-Leffler, preprint.
3. P. Collet and J.-P. Eckmann, *Iterated maps on the interval as dynamical systems*, Progress in Physics **1**, Birkhauser, 1980.
4. M. V. Jacobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Commun. Math. Phys. **81** (1981), 39–88.
5. Y. Kifer, *General random perturbations of hyperbolic and expanding transformations*, J. Analyse Math. **47** (1986), 111–150 (this issue).
6. M. Misiurewicz, *Absolutely continuous measures for certain maps of an interval*, Publ. Math. IHES **53** (1981), 17–51.
7. V. V. Petrov, *Sums of Independent Random Variables*, Springer, Berlin, 1975.

DEPARTMENT OF MATHEMATICS
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CA 91125, USA

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL

(Received November 13, 1985)