Homogeneous and other algebraic dynamical systems and group actions

A. Katok
Penn State University

Functorial constructions

(For a general overview see [10, Sections 1.3, 2.2, 3.4])

- *The restriction* of an action to a subgroup. In the abelian setting the most relevant situation are restrictions of an $\mathbb{R}^k$ action to a connected subgroup isomorphic to $\mathbb{R}^l$ for $1 < l < k$, or to a lattice.

- *Cartesian product* of actions α and β of groups G and H on spaces X and Y correspondingly is the action $\alpha \times \beta$ of $G \times H$ on $X \times Y$ given by

$$\alpha \times \beta(g, h)(x, y) = (\alpha(g)(x), \beta(h)(y)).$$

  Restrictions of the Cartesian product to various subgroups of $G \times H$ are also considered.

  Of particular interest is the *diagonal action*, the restriction of the Cartesian square $\alpha \times \alpha$ to the diagonal subgroup of $G \times G$.

- *Quotient actions* of various kinds, including projections to orbit spaces of finite and other group actions commuting with a given action.

- *Suspension* of a $\mathbb{Z}^k$-action. Let α be a $\mathbb{Z}^k$ action on N. Embed $\mathbb{Z}^k$ as the standard lattice in $\mathbb{R}^k$. $\mathbb{Z}^k$ acts on $\mathbb{R}^k \times N$ by

$$\beta(x, n) = (x - z, z\, n)$$

and form the quotient

$$M = \mathbb{R}^k \times N/\mathbb{Z}^k.$$

Note that the action of $\mathbb{R}^k$ on $\mathbb{R}^k \times N$ by $x\,(y, n) = (x + y, n)$ commutes with the $\mathbb{Z}^k$-action β and therefore descends to M. This $\mathbb{R}^k$-action is called the *suspension* of the $\mathbb{Z}^k$-action.

- *Natural extension* of a $\mathbb{Z}_+^k$ action α on X is a $\mathbb{Z}^k$ action α_e on the space X_P of "pasts" i.e. all maps $p : -\mathbb{Z}_+^k \to X$ such that if $m \in -\mathbb{Z}_+^k$, $n \in \mathbb{Z}_+^k$ and $m + n \in -\mathbb{Z}_+^k$ then

$$p(m+n) = \alpha(n)p(m).$$

The natural extension $\alpha_e : X_P \to X_P$ is defined for any $m \in \mathbb{Z}^k$ as follows: $\mathbb{Z}_+^k$ acts on pasts coordinate-wise by the given action α, $-\mathbb{Z}_+^k$ acts by choosing pre-images from the same past, and the rest of the group is generated by the action of $\mathbb{Z}_+^k \cup -\mathbb{Z}_+^k$.

Notice that when X is a manifold, the space X_P usually is not. An important case is an action $\mathbb{Z}_+^k$ by covering maps: in this case X_P has locally the structure of the product of a Euclidean space and Cantor set; *solenoids* (Section 39) provide typical examples of this situation.

Roots, Lyapunov exponents and Weyl chambers for linear actions

Let ρ be an action of a group A, which may be either $\mathbb{Z}^k$ or $\mathbb{R}^k$, by linear transformations of $\mathbb{R}^m$, or, equivalently, an embedding $\rho : A \to GL(m, \mathbb{R})$. Let $\lambda : A \to \mathbb{C}$ be a *character* or an eigenvalue of the action, i.e. for some vector $v \in \mathbb{R}^m$ and for every $a \in A$,

$$\rho(a)v = \lambda(a)v.$$

The space $\operatorname{Ker}(\rho - \lambda \mathrm{Id})^m \stackrel{\text{def}}{=} R_\lambda$ is the *root space* corresponding to the eigenvalue λ.

$\mathbb{R}^m$ splits into the direct sum of the root spaces corresponding to different real eigenvalues and the real parts of the sums of the roots spaces corresponding to the pairs of complex conjugate eigenvalues. (A version of the Jordan normal form theorem.)

Definition 1 *For an eigenvalue λ let $\chi(\lambda) = \log|\lambda|$. Any such χ is called a* Lyapunov exponent *of the action ρ.*

Let E_χ be the sum of all root spaces R_λ such that $\chi(\lambda) = \chi$. The space E_χ is usually called the Lyapunov space *for the exponent χ.*

The dimension of the Lyapunov space E_χ is the multiplicity *of the Lyapunov exponent χ.*

For a given element of the action the sum of all Lyapunov spaces for the exponents which have positive (corr. negative) values at this element is the *expanding* (or *unstable*) (corr. *contracting* (or *stable*)) space for that element.

If $A = \mathbb{Z}^k$ Lyapunov exponents can be uniquely extended to $\mathbb{R}^k$ so we will always assume that Lyapunov exponents are defined on $\mathbb{R}^k$.

Definition 2 *The kernel of a non-zero Lyapunov exponent is called a* Lyapunov hyperplane*.*

Connected components of the complement to the union of Lyapunov hyperplanes are called the Weyl chambers *for the linear action.*

An element of an action is *regular* if it does not lie on any of the Lyapunov hyperplanes; thus:

Weyl chambers are connected components of the set of regular elements.

A linear action is *hyperbolic* is none of the Lyapunov exponents is identically equal to zero. It is *partially hyperbolic* if there is at least one non-zero Lyapunov exponent.

Algebraic actions

An algebraic action is a non-linear action α (usually on a compact manifold whose infinitesimal behavior can be described by a single linear action called the *linear part* of α. Of particular interest in dynamics are those algebraic actions whose linear part is hyperbolic (*Algebraic Anosov actions*) or partially hyperbolic. All algebraic actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$ are constructed using projections of translations and automorphisms of Lie groups to various coset spaces. Principal classes of algebraic actions:

- Actions of $\mathbb{Z}^k$ by automorphisms of a torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (Generalizations: affine maps and actions on (infra)-nilmanifolds)
- Actions of $\mathbb{R}^k$ by left translations on homogeneous spaces of (semi)-simple Lie groups, such as $SL(n, \mathbb{R})$.

Automorphisms and affine maps on tori and nilmanifolds

An automorphism of the torus $\mathbb{T}^m$ is determined by an $m \times m$ matrix A with integer entries and determinant ± 1. Our standard notation for this automorphism is F_A. Sometimes the group of all such matrices, which is isomorphic to the group of automorphisms of the torus $\mathbb{T}^m$, is denoted by $GL(m, \mathbb{Z})$.

The dual (charcter) group to $\mathbb{T}^m$ is $\mathbb{Z}^m$. The dual to F_A is the automorphism $A^* : \mathbb{Z}^m \to \mathbb{Z}^m$ given by the matrix transposed to A.

Hyperbolicity, ergodicity and Bernoulii property:

Proposition 3 *The following conditions are equivalent:*

1. None of the eigenvalues of the matrix A is a root of unity.
2. Periodic points of F_A are exactly points all of whose coordinates are rational
3. The automorphism F_A is ergodic with respect to Lebesgue measure.
4. Every orbit of the dual map A^*, except that of zero, is infinite.
5. The automorphism F_A is Bernoulli with respect to Lebesgue measure.

Definition 4 *An automorphism of a torus satisfying any of the conditions above is called* ergodic.

All these equivalences except for deducing 5 from any of the other conditions (which relies on *Ornstein Isomorphism Theory*, see [19, 26]) are elementary.

Sketch of proof. The implication $5 \to 3$ are obvious.

Condition 1 implies that the transposed matrix A^t also has no roots of unity among its eigenvalues. Hence all orbits of the dual map A^* on the character group $\mathbb{Z}^m$ other that that of the trivial character, are infinite (condition 4). There are always countably many such orbits. Hence not only F_A is ergodic (condition 3) but the corresponding Koopman operator in $L^2(\mathbb{T}^m)$ has countable Lebesgue spectrum.

Existence of a root of unity among the eigenvalues of A (the negation of 1) and hence also of A^t means that a certain power of A^t has a non-zero invariant vector. This also implies negation of 2 since a power of A (and hence of F_A) has a whole line of fixed points.

The space of all invariant vectors for a power of A^t is rational and hence contains an element with integer coordinates. In other words, a non-trivial character χ is invariant with respect a power of A^*, say $(A^*)^n$ and hence

$$\sum_{i=0}^{n-1} \chi \circ F_A^i$$

is a non-constant invariant function, contradicting 3.

Finally, fixed points of F_A^n are obtained from equations

$$(A^n - \mathrm{Id})x = k$$

for $k \in \mathbb{Z}^m$. Condition 1 implies that for each k solution is unique and can be found from Kramer's rule, hence rational; this implies 2.

Thus we have shown that conditions 1,2,3,4 are equivalent. $\square$

Proposition 5 *Any of conditions 1-5 implies that* the matrix A is partially hyperbolic.

Sketch of proof. Assume first that the matrix A is semisimple (no non-trivial Jordan blocks). If all eigenvalues have absolute value one then $A^{n_k} \to \mathrm{Id}$ for a certain sequence $n_k \to \infty$. But since all powers of A are integer matrices this implies that for a large enough k, $A^{n_k} = \mathrm{Id}$, hence all eigenvalues are roots of unity contradicting 1.

If there are Jordan blocks there is an invariant rational subspace L such that A restricted to L is semisimple. Since L is rational its intersection with the integer lattice is a lattice in L. Hence restriction of A to L is an integer matrix expressed in that basis. Now the previous argument applies. $\square$

Since every ergodic automorphism of the torus is Bernoulli (condition 5) *from the measure theory point of view all ergodic automorphisms are classified by their entropy* which is equal to the *sum of positive Lyapunov characteristic exponents*.

This follows from the Ornstein Isomorphism Theorem [26, Theorem 6.5].

A remark on affine actions: Any affine map whose linear part A is partially hyperbolic has a fixed point and hence is isomorphic to the ergodic automorphism F_A via the translation which takes zero into a fixed point.

For several commuting affine maps the set of fixed points of any of them is invariant under the others. Thus any abelian group of affine maps of a torus which contains an element with partially hyperbolic linear part has a finite orbit and contains a subgroup of finite index which has a fixed point and is hence isomorphic to an action by automorphisms. However, the whole group may not have a fixed point even if all non-zero elements of the action are hyperbolic [11].

Higher rank actions

The genuine higher rank condition: Let α and α' be actions of $\mathbb{Z}^k$ by automorphisms of $\mathbb{T}^m$ and $\mathbb{T}^{m'}$ correspondingly. Then α' is called *an algebraic factor of* α if there exists a surjective homomorphism $h; \mathbb{T}^m \to \mathbb{T}^{m'}$ such that $\alpha' \circ h = \alpha$.

The factor action α' is called a *rank–one factor* of α if $\alpha'(\mathbb{Z}^k)$ has a subgroup of finite index which consists of powers of a single map.

The following two conditions are equivalent [25]:

($\mathcal{R}$) *The action α contains a subgroup ρ, isomorphic to $\mathbb{Z}^2$, which consists of ergodic automorphisms.*

($\mathcal{R}'$) *The action α has no non–trivial rank one algebraic factors.*

Either of these conditions describes the most general "*genuine higher rank*" situation and is sufficient for a number of important rigidity properties.

Irreducibility: An important class of genuinely higher rank actions are those irreducible over $\mathbb{Q}$.

Definition 6 *The action α on $\mathbb{T}^n$ is called irreducible if any nontrivial algebraic factor of α has finite fibres.*

Proposition 7 *Any irreducible over $\mathbb{Q}$ automorphism of a torus is ergodic.*

Proposition 8 *[1] The following conditions are equivalent:*

1. *α is irreducible;*
2. *ρ_α contains a matrix with characteristic polynomial irreducible over $\mathbb{Q}$;*
3. *ρ_α does not have a nontrivial invariant rational subspace or, equivalently, any α–invariant closed subgroup of $\mathbb{T}^n$ is finite.*

Any matrix with irreducible over $\mathbb{Q}$ characteristic polynomial has simple eigenvalues because otherwise the characteristic polynomial and its derivative are not relatively prime and hence the former is reducible via the Euclidean algorithm.

Corollary 9 *Any irreducible free action α of $\mathbb{Z}_+^d$, $d \geq 2$, satisfies condition $(\mathcal{R}')$.*

A rank one algebraic factor has to have fibres of positive dimension. Hence the pre–image of the origin under the factor map is a union of finitely many rational tori of positive dimension and by Proposition 8 α cannot be irreducible.

Irreducible actions and units in number fields: [14, Section 3.3] There are close connections between irreducible actions on $\mathbb{T}^n$ and groups of units in number fields of degree n. In fact, algebraic number theory provides an imporatnt technique for the study of $\mathbb{Z}^k$ actions ay automorphisms of a torus.

Let $A \in GL(n, \mathbb{Z})$ be a matrix with an irreducible characteristic polynomial f and hence distinct eigenvalues. The centralizer of A in $M(n, \mathbb{Q})$ can be identified with the ring of all polynomials in A with rational coefficients modulo the principal ideal generated by the polynomial $f(A)$, and hence with the field $K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of A, by the map

$$\mathcal{G} : p(A) \mapsto p(\lambda) \tag{0.1}$$

with $p \in \mathbb{Q}[x]$. Notice that if $B = p(A)$ is an integer matrix then $\mathcal{G}(B)$ is an algebraic integer, and if $B \in GL(n, \mathbb{Z})$ then $\mathcal{G}(B)$ is an algebraic unit (converse is not necessarily true).

Lemma 10 *The map $\mathcal{G}$ in* (0.1) *is injective.*

Proof. If $\mathcal{G}(p(A)) = 1$ for $p(A) \neq \mathrm{Id}$, then $p(A)$ has 1 as an eigenvalue, and hence has a rational subspace consisting of all invariant vectors. This subspace must be invariant under A which contradicts its irreducibility. □

Denote by $\mathcal{O}_K$ the ring of integers in K, by $\mathcal{U}_K$ the group of units in $\mathcal{O}_K$, by $C(A)$ the centralizer of A in $M(n, \mathbb{Z})$ and by $Z(A)$ the centralizer of A in the group $GL(n, \mathbb{Z})$.

Lemma 11 *$\mathcal{G}(C(A))$ is a ring in K such that $\mathbb{Z}[\lambda] \subset \mathcal{G}(C(A)) \subset \mathcal{O}_K$, and $\mathcal{G}(Z(A)) = \mathcal{U}_K \cap \mathcal{G}(C(A))$.*

Proof. $\mathcal{G}(C(A))$ is a ring because $C(A)$ is a ring. As we pointed out above images of integer matrices are algebraic integers and images of matrices with determinant ± 1 are algebraic units. Hence $\mathcal{G}(C(A)) \subset \mathcal{O}_K$. Finally, for every polynomial p with integer coefficients, $p(A)$ is an integer matrix, hence $\mathbb{Z}[\lambda] \subset \mathcal{G}(C(A))$. □

Notice that $\mathbb{Z}(\lambda)$ is a finite index subring of $\mathcal{O}_K$; hence $\mathcal{G}(C(A))$ has the same property.

Remark *The groups of units in two different rings, say $\mathcal{O}_1 \subset \mathcal{O}_2$, may coincide. Examples can be found in the table of totally real cubic fields [3].*

Proposition 12 *$Z(A)$ is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \times F$ where r_1 is the number the real embeddings, r_2 is the number of pairs of complex conjugate embeddings of the field K into $\mathbb{C}$, and F is a finite cyclic group.*

By Lemma 11, $Z(A)$ is isomorphic to the group of units in the order $\mathcal{G}(C(A))$, so the statement follows from the Dirichlet Unit Theorem ([2], Ch.2, §4.3).

Since $r_1 + 2r_2 = n$, Proposition 12 gives a bound on the rank of an irreducible $\mathbb{Z}^k$ action on $\mathbb{T}^n$.

Conjugacy over $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}$:

Any $\mathbb{Z}^k$ action α by automorphisms of $\mathbb{T}^m$ generated by $F_{A_1}, \ldots, F_{A_k}$ where $A_1, \ldots, A_k$ are integral matrices defines, an embedding $\rho_\alpha : \mathbb{Z}^k \to GL(m, \mathbb{Z})$ by

$$\rho_\alpha^{\mathbf{n}} = A_1^{n_1} \ldots A_d^{n_k},$$

where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^k$.

Actions α and α' are conjugate via an automorphism (*algebraically isomorphic*) if and only if corresponding embeddings ρ_α and ρ'_α are conjugate over $\mathbb{Z}$. This of course implies conjugacy over $\mathbb{Q}$ which is equivalent to the conjugacy over $\mathbb{C}$ and hence, in the irreducible case, is determined by the eigenvalue structure.

The opposite however is not true in general.

The conjugacy over $\mathbb{Z}$ is determined not just by linear algebra as conjugacy over $\mathbb{Q}$ but by the algebraic number theory data. More specifically, it has to do with the *class numbers of the algebraic fields* obtained from adding roots of characteristic polynomials.(see [14, Theorem 4.5])

Algebraic conjugacy and actions on lattices: Every action by automorphisms of a torus has many algebraic factors with finite fibres. These factors are in one–to–one correspondence with lattices $\Gamma \subset \mathbb{R}^m$ which contain the standard lattice $\Gamma_0 = \mathbb{Z}^n$, and which satisfy $\rho_\alpha(\Gamma) \subset \Gamma$.

The factor–action associated with a particular lattice $\Gamma \supset \Gamma_0$ is denoted by α_Γ. In the case of actions by automorphisms such factors are also invertible:

if $\Gamma \supset \Gamma_0$ and $\rho_\alpha(\Gamma) \subset \Gamma$, then $\rho_\alpha(\Gamma) = \Gamma$.

Let $\Gamma \supset \Gamma_0$ be a lattice. Take any basis in Γ and let $S \in GL(n, \mathbb{Q})$ be the matrix which maps the standard basis in Γ_0 to this basis. Then obviously the factor–action α_Γ is equal to the action $\alpha_{S\rho_\alpha S^{-1}}$. In particular, ρ_α and ρ_{α_Γ} are conjugate over $\mathbb{Q}$, although not necessarily over $\mathbb{Z}$.

For any positive integer q, the lattice $\frac{1}{q}\Gamma_0$ is invariant under any automorphism in $GL(n,\mathbb{Z})$ and gives rise to a factor which is conjugate to the initial action: one can set $S=\frac{1}{q}\mathrm{Id}$ and obtains that $\rho_\alpha=\rho_{\alpha_{\frac{1}{q}\Gamma_0}}$. On the other hand, one can find, for any lattice $\Gamma\supset\Gamma_0$, a positive integer q such that $\frac{1}{q}\Gamma_0\supset\Gamma$ (take q the least common multiple of denominators of coordinates for a basis of Γ). Thus $\alpha_{\frac{1}{q}\Gamma_0}$ appears as a factor of α_Γ. To summarize:

Proposition 13 [14, Proposition 4.1] *Let α and α' be $\mathbb{Z}^d$–actions by automorphism of the torus $\mathbb{T}^n$. The following are equivalent.*

1. *ρ_α and $\rho_{\alpha'}$ are conjugate over $\mathbb{Q}$;*
2. *there exists an action α'' such that both α and α' are isomorphic to finite algebraic factors of α'';*
3. *α and α' are weakly algebraically isomorphic, i.e. each of them is isomorphic to a finite algebraic factor of the other.*

Cartan actions: Of particular interest are abelian groups of ergodic automorphisms by $\mathbb{T}^n$ of maximal possible rank $n-1$,

Definition 14 *An action of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$ for $n \geq 3$ by ergodic automorphisms is called a* Cartan action*.*

The following fact easily follows from Proposition 12.

Proposition 15 *Let α be a Cartan action on $\mathbb{T}^n$. Then*

1. *Any element of the action other than identity has real eigenvalues and is hyperbolic and thus Bernoulli*

2. *α is irreducible.*

3. *The centralizer of α is a finite extension of α.*

Lemma 16 *Let A be a hyperbolic matrix in $SL(n, \mathbb{Z})$ with irreducible characteristic polynomial and distinct real eigenvalues. Then every element of the centralizer $Z(A)$ other than $\{\pm \mathrm{Id}\}$ is hyperbolic.*

Proof. Assume that $B \in Z(A)$ is not hyperbolic. As B is simultaneously diagonalizable with A and has real eigenvalues, it has an eigenvalue $+1$ or -1. The corresponding eigenspace is rational and A–invariant. Since A is irreducible, this eigenspace has to coincide with the whole space and hence $B = \pm \mathrm{Id}$. $\square$

Corollary 17 *Cartan actions are exactly the maximal rank irreducible actions corresponding to totally real number fields. The centralizer $Z(\alpha)$ for a Cartan action α is isomorphic to $\mathbb{Z}^{n-1} \times \{\pm \mathrm{Id}\}$. Lyapunov exponents for a Cartan action are simple and Lyapunov hyperplanes are in general position and are* completely irrational, *i.e. none of them contains an integer point.*

Remarkable properties of Cartan actions

- **Global rigidity**: Any Anosov action homotopic to a Cartan action is *differentiabaly conjugate* to it (F. Rodriguez Hertz, preprint).
- **Isomorphism rigidity**: Two Cartan actions are measurably isomorphic only if they are *algebraically isomorphic.*[12, 14].
- **Measure rigidity**: The only ergodic invariant measure for a Cartan action such that some element has positive entropy is *Lebesgue* [16, 18].
- **Rigidity of measurable centralizer**: The centralizer of a Cartan action in the group of Lebesgue measure preserving transformations is a *finite extension* of the action and consists of affine transformations [12, 14].

Example 18 [14, Section 6.3] *Consider two Cartan actions of* $\mathbb{Z}^2$ *on* $\mathbb{T}^3$ *generated by automorphisms* $F_A,\ F_B$ *and* $F_{A'},\ F_{B'}$ *correspondingly, where*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 8 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 8 & 4 \end{pmatrix},$$

and

$$A' = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ -5 & 9 & 2 \end{pmatrix} \quad B' = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ -5 & 9 & 5 \end{pmatrix}.$$

This two actions are isomorphic over $\mathbb{Q}$ and hence by Proposition 13 are algebraic factors of each other with finite fibers. They are however not isomorphic over $\mathbb{Z}$ and hence by the measure rigidity *not measurably isomorphic*.

However every element is Bernoulli (and hence has a huge measurable centralizer) and the entropy structures coincide. This is a remarkable example of a

Rigid construction from soft elements.

Rigidity for genuinely higher rank actions

Let α be a genuinely higher rank action by automophisms of $\mathbb{T}^m$. It has the following properties similar to those of Cartan actions:

- **Local differentiable rigidity**: Any smooth action whose generators are sufficiently close to those of α *differentiabaly conjugate* to α [17, 4, 5].
- **Isomorphism rigidity**: Any action by automorphisms of a torus measurably isomorphic to α is *algebraically isomorphic* to it [12, 14]
- **Measure rigidity**: (Anosov case) The only ergodic invariant measures for α such that some element has positive entropy are *Lebesgue measures on close invariant subgroups* [8].
- **Rigidity of measurable centralizer**: The centralizer of α in the group of Lebesgue measure preserving transformations consists of *affine transformations*[12, 14].

More examples on the torus

Symplectic actions on $\mathbb{T}^4$: The real rank of the group $Sp(4,\mathbb{R})$ of invertible symplectic 4×4 matrices is two. Accordingly, any maximal split Cartan subgroup of $Sp(4,\mathbb{R})$ may intersect the integer lattice $Sp(4,\mathbb{Z})$ by a group of rank at most two. In fact, such an intersection may have rank two and be irreducible over $\mathbb{Q}$. Using it as the linear part one obtains an irreducible $\mathbb{Z}^2$ Anosov action on $\mathbb{T}^4$ by symplectic automorphisms. Let F_A and F_B be generators of such an action. Each of the matrices A and B has two pairs of mutually inverse real eigenvalues. Hence, the four Lyapunov exponents of the action split into two pairs with exponents in each pair differing by sign. Thus there are only two Lyapunov hyperplanes (lines in this case). Geometrically the picture of exponents and Weyl chambers is the same as for the product action generated by $C \times \mathrm{Id}$ and $\mathrm{Id} \times D$ where $C,\ D \in SL(2,\mathbb{Z})$ are hyperbolic matrices.

The difference between the product and the irreducible case is in that the latter satisfies condition ($\mathcal{R}$) while the former does not. Alternatively, one can explain this as follows. The Lyapunov lines in the irreducible case are irrational and in the product case they are simply coordinate axes. If one consider the suspension of the action in the irreducible case *every* one-parameter subgroup of $\mathbb{R}^2$ acts ergodically including those represented by the Lyapunov line. Each of those subgroups acts by isometires along one of the invariant one-dimensional Lyapunov foliations thus providing an essential geometric ingredient for rigidity properties.

Notice that since A is irreducible with real eigenvalues its centralizer has rank three by Proposition 12; thus the $\mathbb{Z}^2$ symplectic action is embedded into an Anosov action of $\mathbb{Z}^3$; a third generator of this action may be chosen to have two pairs of equal eigenvalues whose eigenspaces are spanned by pairs of eigenvectors for the symplectic action with mutually inverse eigenvalues.

Specific examples can be constructed using matrices with recurrent characteristic polynomials which can be easily analyzed explicitly. A more sophisticated version of this method produces Example 20 below.

Genuinely partially hyperbolic actions: A genuinely higher rank action of $\mathbb{Z}^k$ is called *genuinely partially hyperbolic* if it has a zero Lyapunov exponent. In fact, multiplicity of the zero exponent for such an action is always even because the eigenvalues corresponding to the exponent are complex and hence come in conjugate pairs.

Theorem 19 [4, Theorem 3] *Irreducible genuinely partially hyperbolic actions by automorphisms of a torus exist in any even dimension starting from six and not in any other dimension.*

Reducible genuinely partially hyperbolic actions exist in any odd dimension starting from nine.

No genuinely partially hyperbolic actions exist in dimension up to five and seven.

Example 20 *Let* $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2 \\ -2 & 4 & 4 & 0 & 7 & -2 \\ 2 & -6 & -6 & -2 & -10 & 3 \\ -3 & 8 & 9 & 3 & 13 & -4 \\ 4 & -11 & -12 & -3 & -17 & 5 \\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}$. *The* $\mathbb{Z}^2$ *action on* $\mathbb{T}^6$ $F_A^n F_B^m$ *is irreducible and genuinely partially hyperbolic* ([4, Section 6.2])

Natural extensions

Solenoids: The natural extension of a $\mathbb{Z}_+^k$ action by *endomorphisms* of a torus can be identified with a $\mathbb{Z}^k$ action by automorphisms of a *solenoid*, a compact abelian group modeled locally on the product of a Euclidean space (the Archimedean directions) and several additive groups of p-adic integers (the non-Archimedean directions). The Lyapunov exponents accordingly also split into ordinary *Archimedean* and *non-Archimedean*. The weyl chambers analysis extends to this case although the space of the action is no longer a manifold.

We will consider only the simplest and most famous example. For detailed discussion and more elaborate examples see [16].

Example 21 [9, Furstenberg's $\times 2, \times 3$] *The action $E_{2,3}$ of $\mathbb{Z}_+^2$ on the circle generated by the endomorphisms:*

$$E_2 : S^1 \to S^1 \quad x \mapsto 2x, \quad (\mathrm{mod}\ 1)$$

and

$$E_3 : S^1 \to S^1 \quad x \mapsto 3x, \quad (\mathrm{mod}\ 1).$$

The natural extension $S_{2,3}$ of $E_{2,3}$ acts on the dual group of the discrete group $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$. Topologically it is a connected but not locally connected one-dimensional compact, locally modeled on the direct product of $\mathbb{R}$ and the Cantor set. As a group it is an extension of S^1 with the product of dyadic integers and 3-adic integers $\mathbb{Z}_2 \times \mathbb{Z}_3$ in the fiber.

The Lyapunov exponents:

Identify the "time" $\mathbb{Z}^2$ with the integer lattice in the plane $\mathbb{R}^2$ with coordinates s, t. There are three Lyapunov exponents for $S_{2,3}$: the Archimedean

$$t \log 2 + s \log 3$$

and two non-Archmedean:

$$-t \log 2 \quad \text{and} \quad -\mathrm{s} \log 3.$$

This can be seen form the observarion that the multiplication by two acts as in isometry on $\mathbb{Z}_3$ and as a contraction with constant coefficient of contraction $1/2$ on $\mathbb{Z}_2$ and correspondingly the multiplication by three acts as an isometry on $\mathbb{Z}_2$ and as a contraction with coefficient $1/3$ on $\mathbb{Z}_3$.

Lyapunov lines and Weyl chambers:

Thus, there are three Lyapunov lines in the general position

$$t \log 2 + s \log 3 = 0, \quad t = 0 \text{ and } \mathrm{s} = 0.$$

and six Weyl chambers and combinatorially the picture looks exactly the same as for any Cartan action of $\mathbb{Z}^2$ on $\mathbb{T}^3$.

The positive quadrant constitutes one of the six Weyl chambers, namely the one where the Archimedean exponent is positive and two non-Archimedean ones are negative.

A nilpotent example: The simplest non-abelian counterpart of Example 21 appears on a three-dimensional nilmanifold.

Example 22 *Let H be the Heisenberg group of 3×3 upper-diagonal unipotent matrices*
$$N \stackrel{\text{def}}{=} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R},$$
$\Lambda \subset H$ be the subgroup of integer matrices, $\rho_2 : H \to H$ be the following automorphism
$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2x & 4y \\ 0 & 1 & 2z \\ 0 & 0 & 1 \end{pmatrix}.$$
Then $\rho(\Lambda) \subset \Lambda$ and ρ projects to a non-invertible expanding map of the compact nil-manifold $X \stackrel{\text{def}}{=} H/\Lambda$ [13, Section 17.3]*. We similarly define the automorphism $\rho_3 : H \to H$ with 2's and 4 in the above formula replaced by 3's and 9. Projections of ρ_2 and ρ_3 to X define an expanding action of $\mathbb{Z}^2_+$.*

There are two Archimedean Lyapunov exponents in this example. In the notations from the previous subsection they can be expressed by

$$\chi_- = t \log 2 + s \log 3$$

and

$$\chi_+ = t \log 4 + s \log 9;$$

χ_- has multiplicity 2 and χ_+ is simple. The Lyapunov distribution of χ_- is non-integrable. The relation $\chi_+ = 2\chi_-$ is a simple example of a *resonance*.

The projection of this action to the center gives the action from Example 21.

For the theory of actions by automorphisms of general compact abelian groups see [24].

Algebraic $\mathbb{R}^k$-actions

A subgroup Λ of a Lie group H is called a *uniform lattice* if Λ is discrete and cocompact, i.e if there is a compact subset $F \subset H$ such that $F \cdot \Lambda = H$. (There are also *non-uniform lattices* such as $SL(n, \mathbb{Z}) \subset SL(n, \mathbb{R})$)[22, 20].

Let A be a subgroup of a connected Lie group H isomorphic to $\mathbb{R}^k$. The group A acts on the quotient H/Λ by left translations. Suppose C is a compact subgroup of H which commutes with A. Then the $\mathbb{R}^k$-action on H/Λ descends to an action on $C \setminus H/\Lambda$. The general *algebraic* $\mathbb{R}^k$*-action* ρ is a finite factor of such an action.

Let $\mathfrak{c}$ be the Lie algebra of C. The *linear part* of ρ is the representation of $\mathbb{R}^k$ on $\mathfrak{c} \setminus \mathfrak{h}$ induced by the adjoint representation of $\mathbb{R}^k$ on the Lie algebra $\mathfrak{h}$ of H. Let $\mathfrak{a}$ be the Lie algebra of A. The linear part of ρ fixes elements of $\mathfrak{a}$. The factor of the linear part of ρ on $\mathfrak{c} \oplus \mathfrak{a} \setminus \mathfrak{h}$ is the *reduced linear part* of ρ.

An algebraic $\mathbb{R}^k$ action is *partially hyperbolic* if its linear part is partially hyperbolic. Such an action is called *Anosov* if its reduced linear part is hyperbolic.

The *Lyapunov exponents, Lyapunov hyperplanes, Weyl chambers* and *regular elements* for algebraic actions are defined as those for their linear parts.

Since for an $\mathbb{R}^k$ action Lyapunov exponents in $\mathfrak{a}$ are zeroes the multiplicity of the zero exponent for such an action is at least k; it is equal to k if and only if the action is Anosov.

Invariant distributions and their integrability:

Given an algebraic action, root spaces, Lyapunov spaces, and other invariant subspaces of the Lie algebra or its factors extend in the right-invariant way to fields of subspaces (also called distributions) invariant for the action. The terminology for the invariant spaces (such as Lyapunov, stable, etc) extends to those distributions.

Integrability of these distributions is determined by the usual bracket Frobenius criterion. If a distribution is uniquely integrable its integral manifolds form an *invariant homogeneous foliation.*

The following invariant distributions are integrable:

- One-dimensional Lyapunov distributions, i.e. those of *simple* (multiplicity one) Lyapunov exponents;
- Stable and unstable distributions for any element
- Lyapunov distribution for the zero Lyapunov exponent which is usually called the *neutral* distribution for the action.
- Intersections of stable distributions for different elements of an action are also integrable.
- In particular, the smallest subspaces which can be obtained intersections of stable distributions correspond to sums of Lyapunov distributions obtained from all exponents proportional to a given one with positive coefficients of proportionality. These integrable *coarse Lyapunov distributions* play the central role in the rigidity theory.

Lyapunov distributions may not be integrable.

The simplest example of a non-integrable Lyapunov distribution, albeit in the non-invertible case, appears in Example 22 for the "smaller" Lyapunov exponent χ_-. Invertible examples also exist.

Non-integrability of Lyapunov distribution E_χ for algebraic actions may appear only if 2χ is also a Lyapunov exponent for the action, a special case of a *resonance*.

Classes of algebraic $\mathbb{R}^k$ actions [15]

Suspensions: Every $\mathbb{Z}^k$ action α by automorphisms or affine maps of a torus,(and also of a nilmanifold or an infranilmanifold generates an $\mathbb{R}^k$ action via the suspension construction.

The suspension of an Anosov (partially hyperbolic) action of $\mathbb{Z}^k$ is an Anosov (partially hyperbolic) action of $\mathbb{R}^k$.

Suspensions are algebraic actions.

Take $G = \mathbb{R}^k \ltimes \mathbb{R}^m$ (or $G = \mathbb{R}^k \ltimes N$), the semi-direct product of $\mathbb{R}^k$ with $\mathbb{R}^m$ (or a simply connected nilpotent Lie group N). Let $\Lambda \subset G$ be the semi-direct product of the lattice $\mathbb{Z}^k \subset \mathbb{R}^k$ with $\mathbb{Z}^m \subset \mathbb{R}^m$ or Γ, a lattice in N. The action of $\mathbb{R}^k$ on G/Λ by left translation is isomorphic to the suspension of the action α.

Weyl chamber flow: This is a leading class of algebraic Anosov and partially hyperbolic $\mathbb{R}^k$ actions.

Let G be a semisimple connected real Lie group of the noncompact type and of $\mathbb{R}$-rank at least 2. Let A be the connected component of identity of a split Cartan subgroup of G. Suppose Γ is an irreducible torsion-free cocompact lattice in G. The centralizer $Z(A)$ of A splits as a product $Z(A) = M\,A$ where M is compact. Since A commutes with M, the action of A by left translations on G/Γ descends to an A-action on $N \stackrel{\text{def}}{=} M \setminus G/\Gamma$.

This is the *Weyl chamber flow*.

Proposition 23 *Any Weyl chamber flow is an Anosov action.*

Proof Let Σ denote the restricted root system of G. Then the Lie algebra $\mathcal{G}$ of G decomposes

$$\mathcal{G} = \mathcal{M} + \mathcal{A} + \sum_{\alpha \in \Sigma} \mathcal{G}^{\alpha}$$

where g^{α} is the root space of α and $\mathcal{M}$ and $\mathcal{A}$ are the Lie algebras of M and A. Fix an ordering of Σ. If X is any element of the positive Weyl chamber $C \subset \mathcal{A}$ then $\alpha(X)$ is nonzero and real for all $\alpha \in \Sigma$. Hence $\exp X$ acts normally hyperbolically on G with respect to the foliation given by the MA-orbits. $\square$

If the group G is $\mathbb{R}$-*split*, i.e. its real rank equals its complex rank then $M = \{\mathrm{Id}\}$. In this case the Weyl chamber flow acts on G/Γ.

In the non-split case the action of A on G/Γ is a compact group extension of the Weyl chamber flow and hence is partially hyperbolic with the zero Lyapunov exponent of extra multiplicity $\dim M$.

Restrictions of the Weyl chamber flows to lattices in A are important examples of discrete partially hyperbolic algebraic actions. Another interesting class of partially hyperbolic actions consists of restrictions of the Weyl chamber flows to continuous subgroups of A of dimension ≥ 2.

Weyl chamber flow on $SL(n, \mathbb{R})/\Gamma$

Weyl chamber flows on certain factors of $SL(n, \mathbb{R})$ appear in number theoretic problems .

Example 24 *The subgroup $D_n^+ \subset SL(n, \mathbb{R})$ of positive diagonals is the connected component of identity of a maximal Cartan subgroup of $SL(n, \mathbb{R})$. Diagonal entries of $d \in D_n^+$ have the form e^{t_i}, $i = 1, \ldots, n$ where $t_1 + \cdots + t_n = 0$. Thus it is convenient to parametrize D_n^+ by coordinates $t_1, \ldots, t_n$ satisfying the relation $t_1 + \cdots + t_n = 0$.*

For $n = 2$ the group A is one-parameter with diagonal entries e^t, e^{-t} and the action is the *geodesic flow* on the surface of constant negative curvature $M \stackrel{\text{def}}{=} SO(2) \setminus SL(2, \mathbb{R})/\Gamma$.

Lyapunov foliations and the Weyl chambers for the Weyl chamber flow α on $X = SL(n,\mathbb{R})/\Gamma$ [7]:

Let $d(\cdot,\cdot)$ denote a right invariant metric on $SL(n,\mathbb{R})$ and the induced metric on X. A foliation F is isometric under $\alpha^{\mathbf{t}}$ if $d(\alpha^{\mathbf{t}}x, \alpha^{\mathbf{t}}y) = d(x,y)$ whenever $y \in F(x)$. Let $1 \le a, b \le n$ always denote two fixed different indices, and let exp be the exponentiation map for matrices. Define the matrix

$$v_{a,b} = \left(\delta_{(a,b)(i,j)}\right)_{(i,j)},$$

where $\delta_{(a,b)(i,j)}$ is 1 if $(a,b) = (i,j)$ and 0 otherwise. So $v_{a,b}$ has only one nonzero entry, namely, that in row a and column b. With this we define the foliation $F_{a,b}$, for which the leaf

$$F_{a,b}(x) = \left\{\exp(sv_{a,b})x : s \in \mathbb{R}\right\} \tag{0.2}$$

through x consists of all left multiples of x by matrices of the form $\exp(sv_{a,b}) = \mathrm{Id} + sv_{a,b}$.

The foliation $F_{a,b}$ is invariant under α, in fact direct calculation shows

$$\alpha^{\mathbf{t}}\big(\mathrm{Id} + sv_{a,b}\big)x = \big(\mathrm{Id} + se^{t_a - t_b}v_{a,b}\big)\alpha^{\mathbf{t}}x, \tag{0.3}$$

the leaf $F_{a,b}(x)$ is mapped onto $F_{a,b}(\alpha^{\mathbf{t}}x)$ for any $\mathbf{t} \in D_n^+$. Consequently the foliation $F_{a,b}$ is contracted (corr. expanded or neutral) under $\alpha^{\mathbf{t}}$ if $t_a < t_b$ (corr. $t_a > t_b$ or $t_a = t_b$). If the foliation $F_{a,b}$ is neutral under $\alpha^{\mathbf{t}}$, it is in fact isometric under $\alpha^{\mathbf{t}}$.

The leaves of the orbit foliation $O(x) = \{\alpha^{\mathbf{t}}x : \mathbf{t} \in D_n^+\}$ can be described similarly using the matrices

$$u_{a,b} = (\delta_{(a,a)(k,l)} - \delta_{(b,b)(k,l)})_{k,l}.$$

In fact $\exp(u_{a,b}) = \alpha^{\mathbf{t}}$ for some $\mathbf{t} \in D_n^+$.

Clearly the tangent vectors to the leaves in (0.2) for various pairs (a, b) together with the orbit directions form a basis of the tangent space at every $x \in X$.

Proposition 25 *Non-zero Lyapunov exponents for the Weyl chamber flow on* $SL(n,\mathbb{R})/\Gamma$ *are* $t_a - t_b$ *where* $a \neq b$ *and* $1 \leq a, b \leq n$. *Zero Lyapunov exponent comes only from the orbit foliation and hence has multiplicity* $n-1$. *Consequently any matrix* $d \in D_n^+$ *whose elements are pairwise different acts normally hyperbolically on* $SL(n,\mathbb{R})/\Gamma$ *and hence is regular.*

For every $a \neq b$ the equation $t_a = t_b$ defines Lyapunov hyperplane $H_{a,b} \subset D_n^+$. Any elements of this hyperplane acts on the foliation $F_{a,b}$ by isometries . Notice that $H_{a,b} = H_{b,a}$ and hence each of these subgroups acts by isometries on two foliations : $F_{a,b}$ and $F_{b,a}$.

The connected components of

$$A = D_n^+ \setminus \bigcup_{a \neq b} H_{a,b}$$

are the Weyl chambers of the flow α. For every $\mathbf{t} \in A$ only the orbit directions are neutral; hence $\mathbf{t}$ is a regular element.

Let $I = \{(a, b) : a < b\}$, and let M_I be the span of $v_{a,b}$ for $(a, b) \in I$ (in the Lie algebra of $SL(n, \mathbb{R})$). For the invariant foliation F_I the leaf through x is defined by

$$F_I(x) = \big\{\exp(w)x : w \in M_I\big\}. \tag{0.4}$$

Furthermore, there exists a Weyl chamber C, called the *positive Weyl chamber*, such that for every $\mathbf{t} \in C$, the leaf $F_I(x)$ is the unstable manifold for $\alpha^{\mathbf{t}}$. In fact
$C = \{\mathbf{t} \in D_n^+ : t_a > t_b \text{ for all } a < b\}$.

Thus, the picture of Weyl chambers in our sense in this case is *exactly the same as that in the classical sense of the theory of simple Lie groups* This remains true for Weyl chamber flows on factors of other simple real Lie groups.

Using classification of simple real Lie groups one can obtain similarly concrete and detailed pictures in those cases. Again our picture coincides with the classical one.

Example 26 *For the Weyl chamber flow on factors $SL(n, \mathbb{C})$ the Lyapunov hyperplanes and Weyl chambers are the same as for $SL(n, \mathbb{R})$ but the every non-zero exponent has multiplicity 2.*

Unlike suspensions of Cartan actions on the torus where *each Lyapunov foliation appears as the whole isometric foliation for a certain one-parameter subgroup of the action*, here foliations $F_{a,b}$ and $F_{b,a}$ cannot be separated in such a way. The same situation appears for all Weyl Chamber flows due to the symmetry of the root systems.

This fact causes serious difficulties in establishing rigidity properties for Weyl chamber flows and related actions.

Those difficulties are overcome by using *non-commutativity* of the Lyapunov distributions, and, specifically the structure of commutators of the subgroups $\exp(sv_{a,b})$ [7, 21, 6]

References

[1] D. Berend, *Multi-invariant sets on tori*, Trans. Amer. Math. Soc. **280** (1983), 509–532.

[2] Z.I. Borevich and I.R. Shafarevich, *Number Theory*, Academic Press, New York, 1966.

[3] H. Cohen, *A course in computational algebraic number theory* Springer, Berlin–Heidelberg–New York, 1996.

[4] D. Damjanovic and A. Katok, *Local rigidity of partially hyperbolic actions of* $\mathbb{Z}^k$ *and* $\mathbb{R}^k$, $k \geq 2$. *I. KAM method and actions on the Torus*, preprint, 2004.

[5] D. Damjanovic and A. Katok, Local rigidity of actions of higher rank abelian groups and KAM method *ERA–AMS*, 2004, ERA-AMS, **10** 2004, 142–154.

[6] D. Damjanovic and A. Katok, Periodic cycle functionals and cocycle rigidity for certain partially hyperbolic $\mathbb{R}^k$ actions, to appear in Discrete and Continuous Dynamical Systems

[7] M. Einsiedler and A. Katok, *Invariant measures on G/Γ for split simple Lie-groups G*, Comm. Pure Appl. Math **56** (2003), 1184–1221.

[8] M. Einsiedler and E. Lindenstrauss *Rigidity properties of $\mathbb{Z}^d$–actions on tori and solenoids*, ERA–AMS, 2004.

[9] H. Furstenberg *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49.

[10] B. Hasselblatt, A. Katok: *Principal structures*, in Handbook in Dynamical Systems, vol. 1A, Elsevier, 2002, 1-203.

[11] S. Hurder, *Affine Anosov actions* Michigan Math. J. **40** (1993), no. 3, 561–575.

[12] B. Kalinin and A. Katok, *Invariant measures for actions of higher rank abelian groups*, Smooth ergodic theory and its applications (Seattle, WA, 1999), Amer. Math. Soc., Providence, RI, 2001, pp. 593–637.

[13] A. Katok and B. Hasselblatt *Intoduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.

[14] A. Katok, S. Katok and K. Schmidt *Rigidity of measurable structure for $\mathbb{Z}^d$-actions by automorphisms of a torus*, Comm. Math. Helvetici, **77** (2002), 718–745.

[15] A. Katok and R. Spatzier, *First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity*, Inst. Hautes Études Sci. Publ. Math. (1994), **79**, 131–156.

[16] A. Katok and R. Spatzier, *Invariant measures for higher-rank hyperbolic abelian actions*, Ergodic Theory Dynam. Systems **16** (1996), no. 4, 751–778.

[17] A. Katok and R. Spatzier, *Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions*. Tr. Mat. Inst. Steklova **216** (1997), Din. Sist. i Smezhnye Vopr., 292–319.

[18] A. Katok and R. Spatzier, Corrections to: "Invariant measures for higher-rank hyperbolic abelian actions" , Ergodic Theory Dynam. Systems **18** (1998), no. 2, 503–507.

[19] Y. Katznelson *E*rgodic automorphisms on $\mathbb{T}^n$ are Bernoulli shifts Israel J. Math.**10** (1971), 186-195.

[20] D. Kleinbock, N. Shah, A. Starkov, *Dynamics of subgroup actions on homogeneous spaces of Lie groups and applications to number theory* in Handbook in Dynamical Systems, vol. 1A, Elsevier, 2002, 813-930.

[21] E. Lindenstrauss, *Rigidity of multiparameter actions*, Israel Math. J. , to appear.

[22] G.A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, 1991.

[23] D. Rudolph, $\times 2$ *and* $\times 3$ *invariant measures and entropy*, Ergod. Th. and Dynam. Syst. **10** (1990), 395–406

[24] K. Schmidt *Dynamical systems of algebraic origin*, Birkhäuser, Basel, 1995

[25] A. Starkov, *First cohomology group, mixing and minimal sets of commutative group of algebraic action on torus*, J. Math. Sci (New York) **95**, (1999), 2576–2582.

[26] J.-P. Thouvenot, *Entropy, isomorphism and equivalence in ergodic theory,* in Handbook in Dynamical Systems, vol. 1A, Elsevier, 2002, 205-238.