

## SMOOTH MODELS OF THURSTON'S PSEUDO-ANOSOV MAPS

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*Dedicated to the memory of V. M. Alexeyev (1932-1980)*

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### 1. Introduction

Pseudo-Anosov maps were singled out by W. Thurston in connection with the problem of classifying diffeomorphisms of a compact connected  $C^\infty$  surface  $M$  up to isotopy (see [T], [F-L-P]). According to Thurston's classification, every diffeomorphism  $f$  of  $M$  is isotopic to an  $f'$  satisfying one of the following:

- (i)  $f'$  is of finite order and is an isometry with respect to a Riemannian metric of constant curvature on  $M$ ;
- (ii)  $f'$  is a "reducible" diffeomorphism, i. e. it leaves a closed curve (possibly having several components) invariant. (In this case  $f'$  can be further analyzed by cutting along that curve);
- (iii)  $f'$  is a pseudo-Anosov map (cf. paragraph 2).

For every diffeomorphism  $f$  the possibilities of getting an  $f'$  satisfying (i) and (iii) [respectively (ii) and (iii)] are mutually exclusive.

Thurston's pseudo-Anosov maps are homeomorphisms which are  $C^\infty$  diffeomorphisms except at finitely many points (singularities). A pseudo-Anosov map preserves a natural absolutely continuous measure whose density is  $C^\infty$  and positive except at the singularities, at which it vanishes. It is Bernoulli with respect to this measure [F-L-P, paragraphs 9-10]. If  $M$  is a torus, then a pseudo-Anosov map is an Anosov diffeomorphism. As will be observed below (cf. paragraph 2.4), for  $M$  with genus greater than 1, a pseudo-Anosov map cannot be made a diffeomorphism by a coordinate change which is smooth outside the singularities or even outside a sufficiently small neighborhood of the singularities. Thus, in order to find diffeomorphism models for (iii) with the same dynamical properties as pseudo-

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Anosov maps it is necessary to apply a non-trivial constructions which is global in nature. In this paper we will construct, for every pseudo-Anosov map  $f$ , a  $C^\infty$  diffeomorphism  $g$  which is topologically conjugate to  $f$  through a homeomorphism isotopic to the identity and which is Bernoulli with respect to a smooth measure (i.e., one whose density is  $C^\infty$  and positive everywhere). In our paper we will assume as in [F-L-P, paragraphs 9-10] that  $M$  is orientable and without boundary. At the end we will indicate modifications which are needed in the general case.

We are grateful to the referee who made several useful observations and comments (cf. for example corollary at the end of Section 2 and the discussion of the non-orientable case in Section 8) and also pointed out several errors in the original text.

## 2. Definitions and preliminaries

2.1. MEASURED FOLIATIONS. — For each  $r > 0$ , let  $\mathcal{D}_r = \{z : |z| < r\} \subset \mathbb{C}$ .

A *measured foliation* of  $M$  is a foliation  $\mathfrak{F}$  with a finite set of singular points  $x_1, \dots, x_m$  and a transverse measure  $\mu$  given as follows. There is a collection of  $C^\infty$  charts  $(\varphi_i, U_i)$ ,  $i = 1, \dots, L$ ,  $L > m$ , with  $\bigcup_i U_i = M$  such that for each  $i$ ,  $1 \leq i \leq m$ , there is an integer  $p = p(i) \geq 3$  (equal to the number of prongs of the singularity of  $\mathfrak{F}$  at  $x_i$ ) for which the chart  $(\varphi_i, U_i)$  satisfies:

- (i)  $\varphi_i(U_i) = \mathcal{D}_{a_i}$ , for some  $a_i > 0$ ;
- (ii)  $\varphi_i(x_i) = 0$ ;
- (iii) leaves of  $\mathfrak{F}$  get mapped to components of the sets  $\{\operatorname{Im} z^{p/2} = \text{constant}\} \cap \varphi_i(U_i)$ ;
- (iv) on  $U_i$ ,  $\mu$  is given by the image of  $|\operatorname{Im} z^{(p-2)/2} dz|$  and for  $i > m$ , the chart  $(\varphi_i, U_i)$  satisfies:

- (i)  $\varphi_i(U_i) = (0, b_i) \times (0, c_i) \subset \mathbb{R}^2$ , for some  $b_i, c_i > 0$ ;
- (ii) leaves of  $\mathfrak{F}$  get mapped to segments  $\{y = \text{constant}\} \cap \varphi_i(U_i)$ ;
- (iii) on  $U_i$ ,  $\mu$  is given by  $|dy|$ .

Of course, the transverse measures are required to be consistently defined on chart overlaps.

2.2. PSEUDO-ANOSOV MAPS. — Thurston's pseudo-Anosov maps have the following form. There are two measured foliations  $(\mathfrak{F}^s, \mu^s)$  and  $(\mathfrak{F}^u, \mu^u)$  with the same singularities  $x_1, \dots, x_m$  and the same number of prongs  $p = p(i)$  at each  $x_i$ ,  $1 \leq i \leq m$ , which are transversal in the usual sense at nonsingular points and which have  $C^\infty$  charts  $(\varphi_i, U_i)$ ,  $1 \leq i \leq m$ , satisfying:

$$(2.1) \quad \left\{ \begin{array}{l} \text{(i) } \varphi_i(U_i) = \mathcal{D}_{a_i} \text{ for some } a_i > 0, \\ \text{(ii) } \varphi_i(x_i) = 0, \\ \text{(iii) leaves of } \mathfrak{F}^s \text{ get mapped to components of the sets } \{\operatorname{Re} z^{p/2} = \text{constant}\} \cap \mathcal{D}_{a_i}, \\ \text{(iv) leaves of } \mathfrak{F}^u \text{ get mapped to components of the sets } \{\operatorname{Im} z^{p/2} = \text{constant}\} \cap \mathcal{D}_{a_i}, \end{array} \right.$$

and there exists a constant  $\lambda > 1$  such that:

$$(2.2) \quad f(\mathfrak{F}^s, \mu^s) = \left( \mathfrak{F}^s, \frac{1}{\lambda} \mu^s \right) \quad \text{and} \quad f(\mathfrak{F}^u, \mu^u) = (\mathfrak{F}^u, \lambda \mu^u).$$

In fact, the pseudo-Anosov maps in [F-L-P, paragraph 9] are constructed so that:

$$(2.3) \quad \begin{cases} \text{the transverse measures } \mu^s \text{ and } \mu^u \text{ on } U_i \text{ are given by:} \\ |\operatorname{Re} z^{(p-2)/2} dz| \quad \text{and} \quad |\operatorname{Im} z^{(p-2)/2} dz|. \end{cases}$$

respectively. The first examples of maps on the surfaces of genus greater than 1 with properties (2.1) and (2.2) were constructed by T. O'Brien and W. Reddy in 1970 [O-R].

At each singular point  $x_i$ , consider the stable and unstable prongs,

$$P_{i,j}^s = \varphi_i^{-1} \left\{ z = \rho e^{i\tau}: 0 \leq \rho \leq a_i, \tau = \frac{2j+1}{p} \pi \right\}, \\ j=0, 1, \dots, p-1,$$

and:

$$P_{i,j}^u = \varphi_i^{-1} \left\{ z = \rho e^{i\tau}: 0 \leq \rho \leq a_i, \tau = \frac{2j}{p} \pi \right\}, \\ j=0, 1, \dots, p-1,$$

and the stable and unstable sectors:

$$S_{i,j}^s = \varphi_i^{-1} \left\{ z = \rho e^{i\tau}: 0 \leq \rho \leq a_i, \frac{2j-1}{p} \pi \leq \tau \leq \frac{2j+1}{p} \pi \right\}, \\ j=0, \dots, p-1$$

and:

$$S_{i,j}^u = \varphi_i^{-1} \left\{ z = \rho e^{i\tau}: 0 \leq \rho \leq a_i, \frac{2j}{p} \pi \leq \tau \leq \frac{2j+2}{p} \pi \right\}, \\ j=0, 1, \dots, p-1.$$

A singular leaf of  $\mathfrak{F}^s$  [ $\mathfrak{F}^u$ ] is defined to be a singularity  $x_i$  together with the extension along  $\mathfrak{F}^s$  [ $\mathfrak{F}^u$ ] (away from  $x_i$ ) of a stable [unstable] prong at  $x_i$ .

Since  $f$  is a homeomorphism,  $f(x_i) = x_{\sigma(i)}$ ,  $i=1, \dots, m$  where  $\sigma$  is a permutation of  $\{1, \dots, m\}$  such that  $p(i) = p(\sigma(i))$  and  $f$  maps the stable prongs at  $x_i$  into the stable prongs at  $x_{\sigma(i)}$  (provided the  $a_i$ 's are chosen so that  $a_i/\lambda^{2/p} \leq a_{\sigma(i)}$ ). Henceforth (except in paragraph 2.4), we will assume that  $\sigma(i) = i$ ,  $i=1, \dots, m$ , and  $f(P_{i,j}^s) \subset P_{i,j}^s$ ,  $j=0, \dots, p-1$ ,  $i=1, \dots, m$ . (The arguments in the general case are the same, but require

more cumbersome notation.) Now consider the mapping  $\Phi_{i,j}: \phi_i S_{i,j}^s \rightarrow \{z: \operatorname{Re} z \geq 0\}$  given by  $\Phi_{i,j}(z) = (2/p) z^{p/2}$ . Let  $\Phi_{i,j}(z) = w = s_1 + is_2$ . Note that:

$$(2.4) \quad (\Phi_{i,j}^{-1})^* |\operatorname{Re} z^{(p-2)/2} dz| = |ds_1|$$

and:

$$(2.5) \quad (\Phi_{i,j}^{-1})^* |\operatorname{Im} z^{(p-2)/2} dz| = |ds_2|.$$

Thus, in  $S_{i,j}^s \cap f^{-1} S_{i,j}^1$  is given by:

$$(2.6) \quad f = \phi_i^{-1} \Phi_{i,j}^{-1} F \Phi_{i,j} \phi_i,$$

where  $f(w) = F((s_1, s_2)) = (\lambda s_1, (1/\lambda) s_2)$ . It is easy to show that  $f$  is expanding linearly in  $p(\geq 3)$  directions and contracting linearly in  $p(\geq 3)$  directions at  $x_i$ , and consequently  $f$  is not differentiable at  $x_i$ . On  $M \setminus \{x_1, \dots, x_m\}$ ,  $f$  can be constructed to be  $C^\infty$ .

2.3. MARKOV PARTITIONS. — The construction in [F-L-P, paragraph 9] provides a Markov partition for a pseudo-Anosov map  $f$ . It consists of "rectangles"  $R_1, \dots, R_N$ , which are subsets of  $M$  satisfying:

$$(i) \quad \bigcup_{i=1}^N R_i = M.$$

(ii) For each  $i=1, \dots, N$ , there are positive numbers  $l_i, \bar{l}_i$  and a homeomorphism  $\psi_i: [0, l_i] \times [0, \bar{l}_i] \rightarrow R_i$ . If  $x \in R_i$  is a singular point of  $f$  then  $x$  must be one of the corner points, namely,  $\psi_i(0, 0), \psi_i(l_i, 0), \psi_i(0, \bar{l}_i)$  or  $\psi_i(l_i, \bar{l}_i)$ ;  $\psi_i$  is a  $C^\infty$  embedding outside of the set of singularities. Moreover, for each  $t \in [0, l_i], \psi_i(\{t\} \times [0, \bar{l}_i])$  is contained in a leaf of  $\mathfrak{F}^s$  and for each  $t \in [0, \bar{l}_i], \psi_i([0, l_i] \times \{t\})$  is contained in a leaf of  $\mathfrak{F}^u$ . [For  $x \in R_i$  with  $(u, v) = \psi_i^{-1}(x)$ , let  $\mathfrak{F}^s(x, R_i) = \psi_i(\{u\} \times [0, \bar{l}_i])$  and  $\mathfrak{F}^u(x, R_i) = \psi_i([0, l_i] \times \{v\})$ .]

(iii)  $\operatorname{Int} R_i \cap \operatorname{Int} R_j = \emptyset$  for  $i \neq j$ .

(iv) If  $x \in \operatorname{Int} R_i$  and  $f(x) \in \operatorname{Int} R_j$  then:

$$f(\mathfrak{F}^s(x, R_i)) \subset \mathfrak{F}^s(f(x), R_j) \quad \text{and} \quad f^{-1}(\mathfrak{F}^u(f(x), R_j)) \subset \mathfrak{F}^u(x, R_i).$$

(v) If  $x \in \operatorname{Int} R_i$  and  $f(x) \in \operatorname{Int} R_j$ , then:

$$f(\mathfrak{F}^u(x, R_i)) \cap R_j = \mathfrak{F}^u(f(x), R_j) \quad \text{and} \quad f^{-1}(\mathfrak{F}^s(f(x), R_j)) \cap R_i = \mathfrak{F}^s(x, R_i),$$

which means that  $f(R_i)$  goes across  $R_j$  just one time.

In fact, in the construction in [F-L-P], the  $\psi_i$ 's can be chosen so that for each  $(u, v) \in [0, l_i] \times [0, \bar{l}_i]$ ,

$$(2.7) \quad \mu^s(\psi_i([0, u] \times \{v\})) = u$$

and:

$$(2.8) \quad \mu^u(\psi_i(\{u\} \times [0, v])) = v.$$

Also, if we let  $\partial_u R_i = \psi_i([0, l_i] \times \{0, \bar{l}_i\})$  and  $\partial_s R_i = \psi_i(\{0, l_i\} \times [0, \bar{l}_i])$ , then we can get:

$$(2.9) \quad \left\{ \begin{array}{l} \bigcup_{i=1}^N \partial_u R_i = L^u, \\ \text{where:} \\ L^u = \bigcup_{\substack{i=1, \dots, m \\ j=0, \dots, p(i)-1}} L_{i,j}^u, \end{array} \right.$$

with each  $L_{i,j}^u$  being a finite segment (with  $x_i$  as one of its endpoints) of a singular leaf extending  $P_{i,j}^u$ , and similarly for  $\bigcup_{i=1}^N \partial_s R_i$ ,  $L_{i,j}^s$  and  $L^s$ .

The transition matrix  $A^f = (a_{ij})_{1 \leq i, j \leq N}$  for the above Markov partition is given by:

$$(2.10) \quad a_{ij} = \begin{cases} 1 & \text{if } \text{Int } f(R_i) \cap \text{Int } R_j \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The number  $\lambda$  defined in (2.2) is the maximum absolute value of eigenvalues of  $A$ ;  $\log \lambda$  is equal to the topological entropy of  $f$  [F-L-P, paragraph 10].

As is shown in [F-L-P], the partition  $\mathcal{R} = \{R_1, \dots, R_N\}$  generates the Lebesgue sigma field under  $f$  and  $(f, \mathcal{R})$  is a mixing Markov process with respect to the normalization of the invariant measure on  $M$  given locally as a Cartesian product  $\mu^u \times \mu^s$ . Hence, by [0]  $f$  is Bernoulli with respect to this (non-smooth) measure.

#### 2.4. ESSENTIAL NON-SMOOTHNESS OF $f$ AT SINGULARITIES

**PROPOSITION.** — *Let  $f$  be a pseudo-Anosov map on  $M$  of the form described above, where the genus of  $M$  is greater than 1, and let  $g$  be a  $C^1$  diffeomorphism of  $M$ . Then  $f$  cannot be topologically conjugate to  $g$  via a homeomorphism which is a  $C^1$  diffeomorphism except at the singularities of  $f$ .*

*Proof.* — Suppose  $f = hgh^{-1}$ , where  $h$  is a homeomorphism which is a  $C^1$  diffeomorphism except at the singularities of  $f$ . Note that for each positive integer  $k$ ,  $f^k = hg^k h^{-1}$  and  $f^k$  is also a pseudo-Anosov map. Thus by replacing  $f$  by  $f^k$  if necessary, we may again assume that the singularities of  $f$  are fixed under  $f$  and  $f$  maps each stable [unstable] prong to itself. Let  $(\mathfrak{F}^s, \mu^s)$  and  $(\mathfrak{F}^u, \mu^u)$  be the stable and unstable measured foliations for  $f$  with expansion constant  $\lambda > 1$ , as described in paragraph 2.2. Let  $\mathcal{R} = \{R_1, \dots, R_N\}$  be a Markov partition for  $f$  with transition matrix  $A^f = (a_{ij})_{1 \leq i, j \leq N}$  such that the conditions given in paragraph 2.3 are satisfied.

By the Euler-Poincaré index theorem (generalized to foliations) [F-L-P, paragraph 5], the genus of  $M$  being greater than 1 implies that  $(\mathfrak{F}^s, \mu^s)$  and  $(\mathfrak{F}^u, \mu^u)$  have at least one singularity each. Thus  $f$  has at least one singularity.

Let  $y$  be a singularity of  $f$ . Assume that the elements of  $\mathcal{R}$  have been labeled so that  $y$  is a corner point of  $R_1$ . Since  $f$  fixes the stable and unstable prongs at  $y$ , it is easy to see that

$a_{11}=1$ . Also there exists an allowable sequence  $k_1, k_2, \dots, k_l$ ,  $l \geq 3$ , of elements of  $\{1, \dots, N\}$  (i.e.  $a_{k_1 k_2}=1, \dots, a_{k_{l-1} k_l}=1$ ) such that  $k_1=k_l=1$  and  $k_2, \dots, k_{l-1}$  are not equal to 1. For each integer  $q \geq 1$ , let  $y_q$  be the point whose name with respect to the Markov partition  $\mathcal{R}$  is  $\dots b_{-2} b_{-1} b_0 b_1 b_2 \dots$  (i.e.  $\{y_q\} = \bigcap_{n=0}^{\infty} \text{cl} \left[ \text{int} \left( \bigcap_{i=-n}^n f^{-i} R_{b_i} \right) \right]$ ) where:

$$b_i = k_j \quad \text{if } i \equiv j \pmod{(q+l)}, \quad j=1, \dots, l$$

and  $b_i=1$ , otherwise.

Clearly  $y_q$  is a periodic point for  $f$ , and it is easy to check that it cannot be a singular point for  $f$ . Thus, for each  $y_q$ , there exists a neighborhood  $V=V(q)$  of the set of singular points for  $f$  such that  $f^n y_q \notin V$  for each integer  $n$ . Note that at each nonsingular point  $f$  and  $f^{-1}$  have characteristic exponents  $[P] \log \lambda$  and  $-\log \lambda$ . Since  $dh$  and  $dh^{-1}$  are bounded on  $M \setminus V$  and  $hM \setminus hV$ , respectively, it follows that  $g$  and  $g^{-1}$  also have characteristic exponents  $\log \lambda$  and  $-\log \lambda$  at each  $hy_q$ .

We now examine the possibilities for  $dg_{hy}$ . Consider a local trivialization of TM near  $hy$  in which  $B=dg_{hy}$  is a matrix representation in Jordan canonical form.

Case 1 :

$$B = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \quad |\alpha| \leq 1.$$

This is the most complicated case. We will give the argument for this case in full detail and be more sketchy with the other cases.

We have:

$$B^K = \begin{pmatrix} \alpha^K & K \alpha^{K-1} \\ 0 & \alpha^K \end{pmatrix}, \quad K \geq 1.$$

Choose  $\varepsilon > 0$  and  $K \geq 1$  such that:

$$\frac{\log(K+\varepsilon)}{K} < \log \lambda.$$

Then there exists  $\delta > 0$  such that if  $\|B_i - B\| < \delta$  for  $i=0, \dots, K-1$ , then:

$$\|B_{K-1} B_{K-2} \dots B_0 - B^K\| < \varepsilon.$$

Choose a neighborhood  $U$  of  $y$  sufficiently small so that if  $\bar{y} \in U$ , then with respect to the above-chosen coordinate system,

$$\|dg_{h\bar{y}} - B\| < \delta, \quad \|dg_{gh\bar{y}} - B\| < \delta, \dots, \|dg_{g^{K-1}h\bar{y}} - B\| < \delta.$$

Then:

$$\|dg_{h\bar{y}}^K - B^K\| = \|dg_{g^{K-1}h\bar{y}} dg_{g^{K-2}h\bar{y}} \dots dg_{h\bar{y}} - B^K\| < \varepsilon.$$

Thus:

$$\|dg_{hy}^K\| < \|B^K\| + \varepsilon \leq K + \varepsilon.$$

Next choose  $q'$  such that if  $\bar{y} \in \bigcap_{n=0}^{q'} \text{cl} \left[ \text{int} \bigcap_{i=-n}^n f^{-i} R_1 \right]$ , then  $\bar{y} \in U$ . Finally let  $q = 2q' + nK$ , where  $n$  is large. Then:

$$\overline{\lim}_{k \rightarrow \infty} \frac{\log \|dg_{hy_q}^k\|}{k} \leq \frac{\log [(K + \varepsilon)^n \|dg\|^{2q' + l}]}{Kn + 2q' + l}.$$

Note that the limit of the right hand side as  $n \rightarrow \infty$  is  $\log(K + \varepsilon)/K < \log \lambda$ . Thus, if  $n$  is chosen sufficiently large, we have:

$$\overline{\lim}_{k \rightarrow \infty} \frac{\log \|dg_{hy_q}^k\|}{k} < \log \lambda.$$

This implies that the characteristic exponents of  $g$  at  $hy_q$  are both less than  $\log \lambda$ , a contradiction.

Case 2 :

$$B = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \quad |\alpha| > 1.$$

Then  $B^{-1}$  has Jordan form  $\begin{pmatrix} 1/\alpha & 1 \\ 0 & 1/\alpha \end{pmatrix}$ . Thus by the argument for Case 1,  $g^{-1}$  would have both characteristic exponents less than  $\log \lambda$  at  $hy_q$  for sufficiently large  $q$ .

Case 3 :  $B$  has complex eigenvalues  $\alpha, \bar{\alpha}$ . Suppose  $|\alpha| \leq 1$ . Then  $B$  is conjugate to a rotation composed with a contraction by  $|\alpha|$ . Consequently, using the construction described in Case 1 and taking  $q$  sufficiently large, we could get the characteristic exponents of  $g$  at  $hy_q$  to be arbitrarily close to  $\log |\alpha|$ , in particular less than  $\log \lambda$ , a contradiction. The case  $|\alpha| > 1$  is similar.

Case 4 :

$$B = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \quad |\alpha_1| \leq |\alpha_2|.$$

If  $|\alpha_1| \leq 1, |\alpha_2| \leq 1$  or  $|\alpha_1| \geq 1, |\alpha_2| \geq 1$ , then the same type of argument as in Case 3 gives a contradiction. Finally, if  $|\alpha_1| < 1 < |\alpha_2|$ , then according to the Hadamard-Perron Theorem [S, paragraph 1.2], the local stable manifold at  $hy$  is homeomorphic to an interval, which is topologically inconsistent with the existence of at least three stable prongs at  $y$ .

As the referee pointed out, this proposition implies the following corollary.

COROLLARY. — *There exists a neighborhood  $V$  of the singular set such that if  $h$  is a conjugacy of  $f$  to  $C^1$  map  $g$  then  $h$  is not  $C^1$  on  $M \setminus V$ .*

*Proof.* — Since  $h^{-1}gh = f$  then for every  $n$ ,  $h = g^n h f^{-n}$ . If the neighborhood  $V$  is small enough then each point different from a singularity has an iterate under  $f$  which is not in  $M \setminus V$ . Thus  $h$  must be  $C^1$  outside of the singular set which contradicts the proposition.

### 3. Construction of smooth pseudo-Anosov maps

Let  $f$  be a pseudo-Anosov map with expansion constant  $\lambda > 1$ , as discussed in paragraph 2. We now describe the construction of a  $C^\infty$  diffeomorphism  $g$ , which is topologically conjugate to  $f$  through a homeomorphism isotopic to the identity, and which is Bernoulli with respect to an invariant measure given by a smooth positive density.

Our construction consists of a local perturbation of  $f$  in a neighborhood of each singularity. In each stable sector it coincides up to the coordinate change  $\Phi_{ij}$  with the “slowing-down” (n° 2) part of the construction from [K, paragraph 2]. The coordinate change  $\Phi_{ij}$  brings the invariant measure for the slowed-down map back to the Lebesgue measure, thus substituting for the “blowing-up” (n° 3) part of the above-mentioned construction. Henceforth we will refer directly to the equations from [K, paragraph 4] and estimates concerning this construction from [B-F-K, Proposition 2.3].

We should also mention that the proof of Corollary 4.3 (and consequently Proposition 2.1) in [K] is incomplete. For, as M Rees pointed out, the statement from [G] about expansive maps in the closure of Anosov diffeomorphisms is not proved. This gap can be filled by the use of J. Franks' Theorem [F, Prop. 2.1] and the fact that  $\lim_{n \rightarrow \infty} l(g_1^{-n} \gamma) = \infty$  for every stable curve  $\gamma$  (cf. paragraph 5). (This fact is a weaker, non-uniform version of our Corollary 6.2, and it is much easier to prove.)

Indeed, there is a principal difference between the toral case and the pseudo-Anosov case for manifolds of genus greater than 1 because the analog of Franks' theorem is false. Thus, Markov partitions provide a more powerful and more universal (but also more difficult) method of proving the topological conjugacy between the “model” and the perturbed map. The main new technical difficulty in this proof is the establishing of uniform contraction for stable curves. This is done in paragraph 5, which is due completely to the first author.

For  $i=1, \dots, m$ , let  $a'_i = (2/p(i)) a_i^{p(i)/2}$ , and for  $r > 0$  such that  $r \leq a'_i$ , let  $D_r^i = \varphi_i^{-1}(\mathcal{D}_{[(p(i)/2)r]^2 r(i)})$ . Then, in particular,  $D_{a'_i}^i = \varphi_i^{-1} \mathcal{D}_{a_i}$ . Assume that the  $a'_i$ 's were chosen so that  $D_{a'_i}^i \cap D_{a'_j}^j = \emptyset$  for  $i \neq j$ ,  $1 \leq i, j \leq m$ . For  $r > 0$  such that  $r \leq a'_i$  for  $i=1, \dots, m$ , let  $D_r = \bigcup_{i=1}^m D_r^i$ .

Since:

$$\begin{aligned} \Phi_{ij} \varphi_i (D_r^i \cap S_{i,j}^s) &= \mathcal{D}_r \cap \{z : \operatorname{Re} z \geq 0\}, \\ (\Phi_{i,j} \varphi_i)^* |ds_1| &= \mu^s \quad \text{and} \quad (\Phi_{i,j} \varphi_i)^* |ds_2| = \mu^u, \end{aligned}$$

for  $j=0, \dots, p(i)-1$ ,  $i=1, \dots, m$ ,  $r \leq a'_i$ , this means that  $D_r^i$  is the disk of radius  $r$  about  $x_i$  in the metric  $\rho = du^2 + dv^2$  [cf. (2.7), (2.8)]. Let us note that this metric is defined consistently



on the whole manifold  $M$  and it is a Riemannian metric with singularities at the points  $x_1, \dots, x_m$ . We will denote the distance generated by this metric by  $d$  and the length of curves in this metric by  $l$ .

Now choose  $0 < r_6 < r_5 < r_4 < r_3 < r_2 < r_1 < r_0 < r'_3 < r'_2 < r'_1 < r'_0$  such that:

$$(3.1) \quad r_2 < r_0/4.$$

(3.2) For each  $i=1, \dots, m$ ,  $F(\overline{\mathcal{D}_{r'_0}}) \subset \mathcal{D}_{r'_0}$ , where  $F$  is as in paragraph 2.2.

(3.3) For each  $i=1, \dots, m$ ,  $FL^u \cap D_{r'_0} \subset \bigcup_{j=1}^{p(i)} P_{i,j}^u$  and :

$$F^{-1}L^s \cap D_{r'_0} \subset \bigcup_{j=1}^{p(i)} P_{i,j}^s, \text{ where } L^u \text{ and } L^s \text{ are as in (2.9).}$$

$$(3.4) \quad F(\overline{\mathcal{D}_{r_{i+1}}}) \subset \mathcal{D}_{r_i}, \quad i=0, 1, 2, 3, 4, 5 \quad \text{and} \quad F(\overline{\mathcal{D}_{r'_{i+1}}}) \subset \mathcal{D}_{r'_i}, \quad i=0, 1, 2.$$

(It then follows from the definition of  $F$  that the same inclusions will also hold with  $F$  replaced by  $F^{-1}$ .)

(3.5) For some positive integer  $q$ :

$$F^{-q}((\mathcal{D}_{r_n} \setminus \mathcal{D}_{r_1}) \cap \{(s_1, s_2) : |s_2| \geq s_1 \geq 0\}) \subset \mathcal{D}_{r'_0} \setminus \mathcal{D}_{r'_1}.$$

(See Fig. 1.)

(3.6) If  $P$  is the maximum number of prongs at a singularity of  $f$ , then:

$$\left(\frac{p}{2}r_5\right)^{2(p-2)/p} < 1 \quad \text{for } p=3, 4, \dots, P.$$

(3.7) If  $H_1$  and  $H_2$  are branches of hyperbolas given by  $s_1 s_2 = \delta_1$ ,  $s_1 \geq 0$ , and  $s_1 s_2 = \delta_2$ ,  $s_1 \geq 0$ , respectively, which pass through  $\mathcal{D}_{r_0}$  and  $\gamma$  is a connected curve lying between  $H_1$  and  $H_2$  which has a point lying in the region  $(\mathcal{D}_{r'_1} \setminus \mathcal{D}_{r'_2}) \cap \{(s_1, s_2) : s_1 \geq |s_2|\}$  and whose tangent vectors all lie within  $45^\circ$  of the vertical direction in the  $(s_1, s_2)$  coordinate system, then  $\gamma$  lies in  $\mathcal{D}_{r'_0} \setminus \mathcal{D}_{r'_1}$  and:

$$\bar{l}(\gamma) < \bar{d}(H_1 \cap \mathcal{D}_{r_0}, H_2 \cap \mathcal{D}_{r_0}),$$

where  $\bar{l}$  and  $\bar{d}$  denote the usual Cartesian length and distance, respectively, in the  $(s_1, s_2)$  coordinate system.

We now define the "slow-down function"  $\Psi_p$  corresponding to a  $p$ -prong singularity,  $3 \leq p \leq P$ , on the interval  $[0, \infty)$  by:

$$(i) \quad \Psi_p(u) = \left(\frac{p}{2}\right)^{(2p-4)/p} u^{(p-2)/p} \quad \text{for } u \leq r_5^2;$$

$$(ii) \quad \Psi_p \text{ is } C^\infty \text{ except at } 0;$$

$$(iii) \quad \Psi'_p(u) \geq 0 \quad \text{for } u > 0;$$

$$(iv) \quad \Psi_p(u) = 1 \quad \text{for } u \geq r_4^2.$$



(These two definitions coincide on  $\bigcup_{i=1}^m D_{i_0}^i \setminus D_{i_1}^i$ , because in  $D_{i_0}^i \cap S_{i,j}^s$ ,  $f$  is given by  $\varphi_i^{-1} \Phi_{i,j}^{-1} F \Phi_{i,j} \varphi_i(x)$ .)

Obviously,  $g$  is a homeomorphism which is  $C^\infty$  everywhere except possibly at the singularities  $x_1, \dots, x_m$ . In fact, we will see that it is  $C^\infty$  everywhere; moreover, in a neighborhood of each singular point,  $g$  is real-analytic. In order to prove this, we will show that locally the vector field  $\bigcup_{j=0}^{p(i)-1} (\varphi_i^{-1} \Phi_{i,j}^{-1})_* (v_{\Psi_{p(i)}})$ , which generates  $g$ , is Hamiltonian with respect to the volume element generating the Lebesgue measure and its Hamiltonian function is real-analytic.

#### 4. Smoothness of $g$

Fix a singular point  $x_i$  and let  $p = p(i)$ . Consider the  $z = t_1 + it_2 = \rho e^{i\tau}$  coordinate systems on  $D_{r_0}^i$  given by the chart  $(\varphi_i, U_i)$  and the coordinate systems  $w = s_1 + is_2 = re^{i\theta}$  on  $\mathcal{D}_{r_0}$ , where  $\Phi_{i,j}^{-1}, j=0, \dots, p-1$ , is defined on  $\mathcal{D}_{r_0} \cap \{(s_1, s_2) : s_1 \geq 0\}$  by:

$$z = \Phi_{i,j}^{-1}(w) = \left(\frac{p}{2}w\right)^{2/p},$$

or, equivalently,

$$\rho = \left(\frac{p}{2}r\right)^{2/p}, \quad \tau = \frac{2}{p}\theta + \frac{2j\pi}{p} \quad \text{where} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

In each sector  $D_{r_0}^i \cap S_{i,j}^s$ ,  $g$  is given, in the  $z$ -coordinate system, as the time-one map of the vector field  $(\Phi_{i,j}^{-1})_* v_{\Psi_p}$ .

The function  $H$  given by  $H(s_1, s_2) = (\ln \lambda) s_1 s_2$  is Hamiltonian for the vector field  $v$  with respect to the volume element  $ds_1 ds_2$ . Hence  $H$  is also Hamiltonian for  $v_{\Psi_p}$  with respect to  $ds_1 ds_2 / \Psi_p(s_1^2 + s_2^2)$ . Then  $H_1 = H \circ \Phi_{i,j}$  is Hamiltonian for  $(\Phi_{i,j}^{-1})_*(v_{\Psi_p})$  with respect to  $(\Phi_{i,j})_*(ds_1 ds_2 / \Psi_p(s_1^2 + s_2^2))$ . We will show that in  $\mathcal{D}_{r_0}(\Phi_{i,j})^*(ds_1 ds_2 / \Psi_p(s_1^2 + s_2^2))$  is the Lebesgue measure  $\rho d\rho d\tau = dt_1 dt_2$  and  $H_1$  is real analytic in the  $(t_1, t_2)$  coordinate system. It then follows that  $g$  is real analytic in  $D_{r_0}^i$ .

We first compute  $(\Phi_{i,j})^*(ds_1 ds_2 / \Psi_p(s_1^2 + s_2^2))$ . We have:

$$d\rho = \left(\frac{p}{2}\right)^{(2-p)/p} r^{(2-p)/p} dr \quad \text{and} \quad d\tau = \frac{2}{p} d\theta.$$

Thus:

$$\left(\frac{p}{2}\right)^{(p-2)/p} r^{(2p-2)/p} d\rho = r dr \quad \text{and} \quad \frac{p}{2} d\tau = d\theta.$$

Hence:

$$\begin{aligned} (4.1) \quad (\Phi_{i,j})^* \left( \frac{ds_1 ds_2}{\Psi_p(s_1^2 + s_2^2)} \right) &= (\Phi_{i,j})^* \left( \frac{r dr d\theta}{\Psi_p(r^2)} \right) \\ &= \frac{((p/2)^{(p-2)/p} r^{(2p-2)/p} d\rho) ((p/2) d\tau)}{(p/2)^{(2p-4)/p} r^{(2p-4)/p}} = \left(\frac{p}{2}\right)^{2/p} r^{2/p} d\rho d\tau = \rho d\rho d\tau. \end{aligned}$$

Next we compute  $H_1$ . In polar coordinates  $H$  is given by:

$$H(re^{i\theta}) = (\ln \lambda) r \cos \theta r \sin \theta = (\ln \lambda) \frac{r^2}{2} \sin 2\theta$$

and:

$$\Phi_{i,j}(\rho e^{i\tau}) = \left(\frac{2}{p}\right) \rho^{p/2} e^{ip\tau/2}.$$

Thus we have:

$$\begin{aligned} H_1(t_1, t_2) &= H_1(\rho e^{i\tau}) = (\ln \lambda) \frac{(2\rho^{p/2}/p)^2}{2} \sin 2(p\tau/2) \\ &= \left(\frac{2\ln \lambda}{p^2}\right) \rho^p \sin p\tau = \left(\frac{2\ln \lambda}{p^2}\right) \operatorname{Im}(t_1 + it_2)^p, \end{aligned}$$

which is a polynomial in  $t_1$  and  $t_2$ , and hence real analytic.

### 5. Uniform Contraction Lemma

Let  $(\eta_1, \eta_2)$  denote the natural coordinates in each tangent space at a point of the  $(s_1, s_2)$ -plane. Furthermore, let  $\mathcal{K}_{(\eta_1, \eta_2)}^+[\mathcal{K}_{(\eta_1, \eta_2)}^-]$  be the closed cone in this tangent space around the  $\eta_1 - |\eta_2|$  axis bounded by the lines  $\eta_1 = \pm \eta_2$ .

Define a coordinate system  $(\xi_1, \xi_2)$  in each tangent space  $T_x M$ ,  $x \in M \setminus \{x_1, \dots, x_m\}$  as follows. Take a system of  $C^\infty$  coordinate charts  $(\varphi_i, U_i)$ ,  $i = 1, \dots, L$ ,  $L > m$ , on  $M$  such that  $\bigcup_i U_i = M$  and for  $i = 1, \dots, m$ ,  $(\varphi_i, U_i)$  satisfies (2.1) (i)-(iv) and (2.3) and for  $i > m$ ,  $(\varphi_i, U_i)$  satisfies:

$$(5.1) \quad \varphi_i(U_i) = (0, b_i) \times (0, c_i) \subset (s_1, s_2)\text{-plane},$$

for some  $b_i, c_i > 0$ ;

$$(5.2) \quad \text{leaves of } \mathfrak{F}^s \text{ get mapped to segments:}$$

$$\{s_1 = \text{constant}\} \cap \varphi_i(U_i);$$

$$(5.3) \quad \text{leaves of } \mathfrak{F}^u \text{ get mapped to segments:}$$

$$\{s_2 = \text{constant}\} \cap \varphi_i(U_i);$$

$$(5.4) \quad \text{on } U_i, \mu^s \text{ and } \mu^u \text{ are given by } |ds_1| \text{ and } |ds_2|, \text{ respectively.}$$

Then in each  $U_i$ ,  $i > m$ , let  $(\xi_1, \xi_2) = d\varphi_i^{-1}(\eta_1, \eta_2)$ , and in the  $j$ -th sector  $S_{i,j}^s$  at  $x_i$ ,  $i = 1, \dots, m$ ,  $j = 0, \dots, p(i) - 1$ , let  $(\xi_1, \xi_2) = (d\varphi_i^{-1} \Phi_{ij}^{-1})(\eta_1, \eta_2)$ . Since  $(\varphi_i)^*$ ,  $i > m$ , and  $(\Phi_{ij} \varphi_i)^*$ ,  $i = 1, \dots, m$ ,  $j = 0, \dots, p(i) - 1$ , each maps  $|ds_1|$  and  $|ds_2|$  to  $\mu^s$  and  $\mu^u$ ,

respectively, we have  $\mu^s(\xi_1, \xi_2) = |\xi_1|$  and  $\mu^u(\xi_1, \xi_2) = |\xi_2|$ . Thus, on chart overlaps the  $(\xi_1, \xi_2)$  coordinate systems are consistently defined, up to changes in sign of each of the coordinates. Also, because:

$$f(\mathfrak{F}^s, \mu^s) = \left( \mathfrak{F}^s, \frac{1}{\lambda} \mu^s \right) \quad \text{and} \quad f(\mathfrak{F}^u, \mu^u) = (\mathfrak{F}^u, \lambda \mu^u),$$

we have  $f_*(\xi_1, \xi_2) = (\pm \lambda \xi_1, \pm (1/\lambda) \xi_2)$ .

For  $x \in M \setminus \{x_1, \dots, x_m\}$  let  $E_{x,f}^u$  and  $E_{x,f}^s$  be the subspaces of  $T_x M$  which are tangent to  $\mathfrak{F}^u$  and  $\mathfrak{F}^s$ , respectively, at  $x$ , and let:

$$K_x^+ = \{(\xi_1, \xi_2) \in T_x M : |\xi_2| \leq |\xi_1|\}$$

and:

$$K_x^- = \{(\xi_1, \xi_2) \in T_x M : |\xi_1| \leq |\xi_2|\}.$$

PROPOSITION 5.1. — For  $x \in M \setminus \{x_1, \dots, x_m\}$ , the families of cones  $K_x^+$  and  $K_x^-$  are semi-invariant, i. e.:

$$dg(K_x^+) \subset K_{gx}^+ \quad \text{and} \quad dg^{-1}(K_x^-) \subset K_{g^{-1}x}^-.$$

For each  $x \in M \setminus \{x_1, \dots, x_m\}$ , the intersections:

$$(5.5) \quad E_{x,g}^u = \bigcap_{n \geq 0} dg^n K_{g^{-n}x}^+ \quad \text{and} \quad E_{x,g}^s = \bigcap_{n \leq 0} dg^n K_{g^{-n}x}^-$$

are one-dimensional subspaces of  $T_x M$ .

(For  $x \in \{x_1, \dots, x_m\}$ , we simply define  $E_{x,g}^s[E_{x,g}^u]$  to be the set of all rays tangent to the stable [unstable] prongs.)

Proof. — The differential equations for  $(\xi_1, \xi_2)$  under the flow given by the vector field  $(\varphi_i^{-1} \Phi_{i,j}^{-1}) * v_{\Psi_{p(i)}}$ , which generates  $g$  in the sector  $S_{i,j}^s$  at  $x_i$ , written in the  $(s_1, s_2) = \Phi_{i,j}(t_1, t_2)$  coordinate system, are identical with those given in (4.1) of [K] (making the notational substitution of  $\Psi_p$  for  $\Psi_\tau$ ). Since we also have  $dg(\xi_1, \xi_2) = df(\xi_1, \xi_2) = (\pm \lambda \xi_1, \pm (1/\lambda) \xi_2)$  at points outside  $\bigcup_{i=1}^m D_i^s$ , which is the only other fact needed in the proof of the analogous

Proposition 4.1 in [K], our proposition follows.  $\square$

As in [K],  $E_{x,g}^u = E_{x,f}^u$  for  $x \in \bigcup_{i,j} P_{i,j}^u \setminus \{x_1, \dots, x_m\}$  and  $E_{x,g}^s = E_{x,f}^s$  for  $x \in \bigcup_{i,j} P_{i,j}^s \setminus \{x_1, \dots, x_m\}$ . Now let  $L_{i,j}^u$  be given as in (2.9). We claim that  $E_{x,g}^u = E_{x,f}^u$  for  $x \in L_{i,j}^u \setminus \{x_1, \dots, x_m\}$ . If  $L_{i,j}^u \subseteq P_{i,j}^u$  this is clear; otherwise take the first positive integer  $n$  such that  $f^n(P_{i,j}^u) \supseteq L_{i,j}^u$ . Note that it follows from (3.3) that  $f^n(P_{i,j}^u) = g^n(P_{i,j}^u)$ , which together with the  $f_*$ - and  $g_*$ -invariance of the line fields  $E_{x,f}^u$  and  $E_{x,g}^u$ , respectively, establishes the claim, i. e., by (2.9), each vector tangent to a  $\partial_u R_j$ ,  $j = 1, \dots, N$ , lies in  $E_{x,g}^u$ . Similarly for  $\partial_s R_j$  and  $E_{x,g}^s$ .

We will call a curve stable [unstable] if it is connected and its tangent vector at each point  $x$  on it belongs to  $E_{x,g}^s \setminus \{0\}$  [ $E_{x,g}^u \setminus \{0\}$ ]. For convenience we also assume that stable curves do not contain  $x_1, \dots, x_m$  as interior points. Since the same simple argument as in [K, Corollary 4.1] shows that  $E_{x,g}^s$  and  $E_{x,g}^u$  depend continuously on  $x$  for  $x \in M \setminus \{x_1, \dots, x_m\}$ , we can conclude from the existence theorems for differential equations that for each  $x \in M \setminus \{x_1, \dots, x_m\}$ , there exists at least one stable [unstable] curve through  $x$ .

Uniqueness of such stable [unstable] curves will be shown in paragraph 6. In fact this will follow from the Uniform Contraction Lemma below, which is also crucial for establishing the topological conjugacy between  $f$  and  $g$ .

We now proceed to establish some properties of stable curves which we need to prove the Lemma.

PROPOSITION 5.2. — Suppose  $\gamma$  is a stable curve lying in the domain  $D_{a_i}^i$  of a singular chart  $\varphi_i$ . Then the following properties hold:

(5.6) Cone condition for stable curves. For  $j=0, \dots, p(i)-1$  the tangent vector to  $\Phi_{i,j}\varphi_i(\gamma)$  at any point  $(s_1, s_2) \in \Phi_{i,j}\varphi_i(\gamma)$  lies in  $\mathcal{K}_{(s_1, s_2)}^-$ .

(5.7) If  $\gamma$  is not equal to a segment of a stable prong, then  $\gamma$  is contained in the interior of some  $S_{i,j}^s$ . (In any case,  $\gamma$  lies in a single  $S_{i,j}^s$ , which is in the domain of  $\Phi_{i,j}\varphi_i$ . Thus we will often work with  $\Phi_{i,j}\varphi_i(\gamma)$  instead of  $\gamma$ .)

(5.8) If  $s \in \Phi_{i,j}\varphi_i(\gamma)$  where  $s=(s_1, s_2)$ ,  $s_1 > 0$ ,  $s_2 > 0$ ,  $s_1 s_2 = \delta$ , then the tangent vector to  $\Phi_{i,j}\varphi_i(\gamma)$  at  $x$  lies in  $[(\text{top half of } \mathcal{K}_s^-) \cap (\text{vectors pointing into the region } s_1 s_2 > \delta)] \cup [(\text{bottom half of } \mathcal{K}_s^-) \cap (\text{vectors pointing into the region } s_1 s_2 < \delta)]$ . Analogous statements hold for  $s$  in other quadrants.

*Proof.* — Property (5.6) is obvious from the definition of a stable curve.

Suppose (5.7) is false. Then there exists a stable sector  $S_{i,j}^s$  such that  $\gamma$  intersects one of the stable prongs bounding  $S_{i,j}^s$ , but also contains points in  $\text{int } S_{i,j}^s$ . Then applying large positive powers of  $G_{p(i)}$  to  $\Phi_{i,j}\varphi_i(\gamma)$  gives curves which have a point on the  $s_2$ -axis arbitrarily close to  $(0, 0)$  and a point arbitrarily close to  $(r_0, 0)$ . But such curves cannot satisfy the cone condition, a contradiction. Hence (5.7) holds.

Finally let us establish (5.8). Suppose the tangent vector  $\eta$  to  $\Phi_{i,j}\varphi_i(\gamma)$  at  $s$  lies in  $(\text{top half of } \mathcal{K}_s^-) \cap (\text{vectors pointing into the region } s_1 s_2 < \delta \text{ or tangent to } s_1 s_2 = \delta)$ . Then under the flow of  $v_{\Psi_{p(i)}}$ ,  $\eta$  is transformed into a vector in  $(\text{top half of } \mathcal{K}_y^-) \cap (\text{vectors pointing into the region } s_1 s_2 < \delta \text{ or tangent to } s_1 s_2 = \delta)$ , where  $y$  is a point on  $s_1 s_2 = \delta$ ,  $s_1 > s_2 > 0$ . But for such a point  $y$  this intersection is empty. Hence  $\eta$  cannot lie in  $(\text{top half of } \mathcal{K}_s^-) \cap (\text{vectors pointing into the region } s_1 s_2 < \delta \text{ or tangent to } s_1 s_2 = \delta)$ . By reversing the time parameter of  $\gamma$ , it follows that  $\eta$  cannot lie in  $(\text{bottom half of } \mathcal{K}_s^-) \cap (\text{vectors pointing into the region } s_1 s_2 > \delta \text{ or tangent to } s_1 s_2 = \delta)$ . Thus (5.8) holds.  $\square$

For each singular chart  $(\varphi_i, D_{a_i}^i)$  satisfying (2.1) (i)-(iv) and (2.3), we have  $l(\gamma \cap D_{a_i}^i \cap S_{i,j}^s) = \bar{l}(\Phi_{i,j}\varphi_i(\gamma \cap D_{a_i}^i \cap S_{i,j}^s))$  and for each chart  $(\varphi_i, U_i)$  satisfying (5.1)-(5.4), we have  $l(\gamma \cap U_i) = \bar{l}(\varphi_i(\gamma \cap U_i))$ . (Recall that  $l$  is the length generated by the metric  $\rho = du^2 + dv^2$  (cf. paragraph 3) and  $\bar{l}$  is the usual Cartesian length.)

UNIFORM CONTRACTION LEMMA. — *There exists  $\tau > 0$  with the following property: for every  $\varepsilon > 0$  there exists  $\mathcal{N} = \mathcal{N}(\varepsilon)$  such that if  $\gamma$  is a stable curve with  $l(\gamma) \leq \tau$  and  $n \geq \mathcal{N}$ , then  $l(g^n \gamma) < \varepsilon$ .*

*Remark.* — Clearly if there is some  $\tau > 0$  for which the lemma is true, then it is true for every  $\tau > 0$ . We will simply choose a  $\tau$  which is convenient for the proof.

*Set up for proof of Uniform Contraction Lemma.* — Let:

$$J = \max(\sup_{x \in M} \|dg_x\|, \sup_{x \in M} \|dg_x^{-1}\|).$$

By proposition 2.3 in [B-F-K], there exist constants  $C_0 > 1$ ,  $0 < K_0 < 1$  such that:

$$(5.9) \quad \|(dg_{D_{r_0}^c})^n e_s\| \leq C_0 K_0^n \|e_s\|, \quad n \geq 0$$

for  $e_s \in E_{x, g}^s$ ,  $x \in D_{r_0}^c$  such that  $g_{D_{r_0}^c}^n x$  is defined. Here we take  $dg_A$ ,  $A \subset M$ , to be the map:

$$dg_A: TA \rightarrow TA \quad \text{by} \quad (dg_A)_x = (dg^{n_x})_x,$$

where  $n_x$  is the first return time of  $x$  to  $A$ . As in paragraph 3, let  $P$  be the largest number of prongs of any singularity of the  $(\mathfrak{F}^u, \mu^u)$  foliation.

(5.10) Let  $a$  be an integer such that for each  $p$ ;  $3 \leq p \leq \underline{P}$ , and for  $x \in \mathcal{D}_{r_2} \setminus G^{-1} \mathcal{D}_{r_2}$ , there exists an integer  $a'$ ,  $0 < a' \leq a$ , such that:

$$G_p^{a'}(x) \in (\mathcal{D}_{r_1} \setminus \mathcal{D}_{r_2}) \cap \{(s_1, s_2): s_1 > |s_2|\}.$$

Let  $\mathcal{B}_0$  be the region bounded by  $\{(s_1, s_2): s_1 = 0\}$ ,

$$\partial \mathcal{D}_{r_0} \cap \{(s_1, s_2): |s_2| \geq s_1 \geq 0\},$$

$$\partial \mathcal{D}_{r_0} \cap \{(s_1, s_2): s_1 \geq |s_2|\},$$

and the branches of hyperbolas  $s_1 s_2 = r_0^2/2$ ,  $s_1 \geq 0$  and:

$$s_1 s_2 = \frac{-r_0^2}{2}, \quad s_1 \geq 0.$$

(5.11) Choose  $\tau'$  such that:

$$0 < \tau' < \min\left(1, r'_0 - r_0, r_0 - r_1, r_1 - r_2, r_2 - r_3, \frac{\sqrt{2}}{4} r_2\right).$$

(5.12) Then if  $\delta_3 > 0$  is chosen sufficiently small and  $\mathcal{B}_3$  is the region bounded by  $\{(s_1, s_2): s_1 = 0\}$ ,  $\partial \mathcal{D}_{r_0}$ , and the branches of the hyperbolas  $s_1 s_2 = \delta_3$ ,  $s_1 \geq 0$  and  $s_1 s_2 = -\delta_3$ ,  $s_1 \geq 0$ , we have:

$$(5.13) \quad \left\{ \begin{array}{l} \tau' < \bar{d}(\mathcal{B}_3 \cap (\mathcal{D}_{r_1} \setminus \mathcal{D}_{r_2}) \cap \{(s_1, s_2): |s_2| \geq s_1\}, \\ \{(s_1, s_2): s_1 \geq |s_2|\}). \end{array} \right.$$

Let  $\tau$  be such that:

$$(5.14) \quad 0 < \tau < \frac{\tau'}{C_0 J^{\max(a, q+2)}},$$

where  $q$  is as in (3.5).

(5.15) Choose  $\varepsilon'$  such that  $0 < \varepsilon' < \min(\varepsilon, \tau')/10$ .

(5.16) Choose a positive integer  $n_0$  such that  $G_p^{n_0}(0, r'_0)$  has  $s_2$  coordinate less than  $\varepsilon'$  for  $3 \leq p \leq P$ . Let  $\mathcal{B}_1$  be the region bounded by  $\{(s_1, s_2); s_1 = 0\}$ ,

$$\begin{aligned} \partial \mathcal{D}_{r'_0} \cap \{(s_1, s_2): |s_2| \geq s_1 \geq 0\}, \\ \partial \mathcal{D}_{r_0} \cap \{(s_1, s_2): s_1 \geq |s_2|\}, \end{aligned}$$

and the branches of hyperbolas  $s_1 s_2 = \delta_1, s_1 \geq 0$  and  $s_1 s_2 = -\delta_1, s_1 \geq 0$ , where  $\delta_1$  is chosen so that:

$$0 < \delta_1 < \min\left(\frac{r_2^2}{2}, (\varepsilon')^2\right)$$

and for any point  $(s_1, s_2) \in \mathcal{B}_1$  with  $|s_2| \geq s_1$ , we have:

$$(5.17) \quad \bar{d}(G_p^n(s_1, s_2), G_p^n(0, s_2)) < \varepsilon' \quad \text{for } 3 \leq p \leq P, \quad 0 \leq n \leq n_0.$$

(5.18) Let  $b$  be a positive integer such that for each  $(s_1, s_2) \in \mathcal{D}_{r_0} \setminus \mathcal{B}_1$  and each  $p$ ,  $3 \leq p \leq P$ , there exists an integer  $n$ ,  $0 < n \leq b$  such that  $G_p^{-n}(s_1, s_2) \in \mathcal{D}_{r_0}^c$ .

Choose  $\xi'$  such that:

$$(5.19) \quad \begin{cases} 0 < \xi' < \bar{d}(\mathcal{B}_1 \cap (\mathcal{D}_{r_1} \setminus \mathcal{D}_{r_2}) \cap \{(s_1, s_2): |s_2| \geq s_1\}, \\ \{(s_1, s_2): s_1 \geq |s_2|\}) \end{cases}$$

and if  $\delta_3$  is chosen sufficiently small and  $\mathcal{B}_3$  is as in (5.12), then:

$$(5.20) \quad \xi' < \bar{d}(\mathcal{B}_3 \cap \mathcal{D}_{r_1}, \mathcal{B}_1^c).$$

Let  $\mathcal{B} = \mathcal{B}_3 \cap \mathcal{D}_{r_2}$ .

Choose  $\varepsilon''$  such that:

$$(5.21) \quad 0 < \varepsilon'' < \frac{\min(\varepsilon', \xi', \tau)}{27 C_0 J^{\max(a+b, q+1)}}.$$

(Since  $C_0 \geq 1$  and  $J \geq 1$  we have  $\varepsilon'' < \varepsilon'$  and  $\tau < \tau'$ .)

(5.22) Choose a positive integer  $n_1$  so that  $G_p^{n_1}(0, r_0)$  has  $s_2$  coordinate less than  $\varepsilon''$  for  $3 \leq p \leq P$ . Let  $\mathcal{B}_2$  be the region bounded by  $\{(s_1, s_2): s_1 = 0\}$ ,  $\partial \mathcal{D}_{r_0}$ , and the branches of



hyperbolas  $s_1 s_2 = \delta_2$ ,  $s_1 \geq 0$  and  $s_1 s_2 = -\delta_2$ ,  $s_1 \leq 0$ , where  $\delta_2$  is chosen so that  $0 < \delta_2 < \min(\delta_1, (\varepsilon'')^2)$  and for any point  $(s_1, s_2) \in \mathcal{B}_2$  with  $|s_2| \geq s_1$ , we have:

$$(5.23) \quad \bar{d}(G_p^n(s_1, s_2), G_p^n(0, s_2)) < \varepsilon'' \quad \text{for } 3 \leq p \leq P, \quad 0 \leq n \leq n_1.$$

Now fix  $\delta_3$  such that  $0 < \delta_3 < \delta_2$  and (5.13) and (5.20) hold.

Note that the region  $\mathcal{A} \cap \{(s_1, s_2): s_2 \geq 0\}$  is symmetric with respect to the line  $s_1 = s_2$  for  $\mathcal{A} = \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}$  and is not for  $\mathcal{A} = \mathcal{B}_0, \mathcal{B}_1$  (see Fig. 2). Let  $B \subset M$  be the union over the singular points and stable sectors of the inverse images of  $\mathcal{B}$  under the corresponding charts  $\Phi_{ij}\varphi_i$ . Define  $B_0, B_1, B_2, B_3$  similarly. Again, by Proposition 2.3 in [B-F-K], there exist constants  $C > 1$ ,  $0 < K < 1$  such that:

$$(5.24) \quad \|(Dg_{B^c})^n e_s\| \leq CK^n \|e_s\|, \quad n \geq 0,$$

for  $e_s \in E_{x,g}^s$ ,  $x \in B^c$  such that  $g_{B^c}^n x$  is defined.

$$(5.25) \quad \text{Let } \xi = \bar{d}(\mathcal{B}_3 \cap \mathcal{D}_{r_1}, \mathcal{B}_2^c).$$

Choose a positive integer  $n_2$  such that:

$$(5.26) \quad CK^{n_2} < \min(\xi, \varepsilon'').$$

$$(5.27) \quad \text{Finally choose } \mathcal{N} > (n_2 + 1) \max(n_1 + 1, n_2 + 1).$$

Keeping the notation in the above set-up, we now formulate and prove Lemmas 5.3-5.7, which will be used in the proof of the Uniform Contraction Lemma.

LEMMA 5.3. — Any curve in the  $(s_1, s_2)$ -plane all of whose tangent vectors are within  $45^\circ$  of being vertical cannot intersect both  $\mathcal{D}_{r_2}$  and  $\mathcal{D}_{r_0}^c \cap \{(s_1, s_2): s_1 \geq |s_2|, |s_1 s_2| \leq \delta_1\}$ .

Proof. — Suppose  $(c_1, c_2) \in \mathcal{D}_{r_2}$  and  $(d_1, d_2) \in \mathcal{D}_{r_0}^c \cap \{(s_1, s_2): s_1 \geq |s_2|, |s_1 s_2| \leq \delta_1\}$ . Since  $\delta_1 < r_2^2/2$ ,  $|d_2| < r_2/\sqrt{2}$ . Then since  $d_1^2 + d_2^2 \geq r_0^2$  and by (3.1)  $r_0 > 4r_2$ , we have:

$$d_1 > \sqrt{r_0^2 - \frac{r_2^2}{2}} > \sqrt{16r_2^2 - \frac{r_2^2}{2}} > 3r_2.$$

Also,  $|c_1| \leq r_2$  and  $|c_2| \leq r_2$ . Thus,  $|d_2 - c_2| < 2r_2$  and  $|c_1 - d_1| > 2r_2$ . Hence:

$$\left| \frac{d_2 - c_2}{d_1 - c_1} \right| < 1.$$

The lemma now follows from the mean-value theorem.  $\square$

LEMMA 5.4. — Let  $l = 0, 1, 2, 3$ . Suppose  $\gamma$  is a stable curve contained in  $B_l$  and  $g\gamma, g^2\gamma, \dots, g^k\gamma$  intersect  $B$ . Then  $\gamma, g\gamma, \dots, g^k\gamma$  all lie in the same component of  $B_l$ .

Proof. — We give the argument for  $B_1$ . That for  $B_2$  and  $B_3$  is identical, while that for  $B_0$  requires one small change which we mention below. Since  $\gamma$  is connected, it lies in a single component of  $B_1$ . Consequently, it is contained in some  $D_{r_0}^i \subset D_{a_i}^i$ , the domain of the

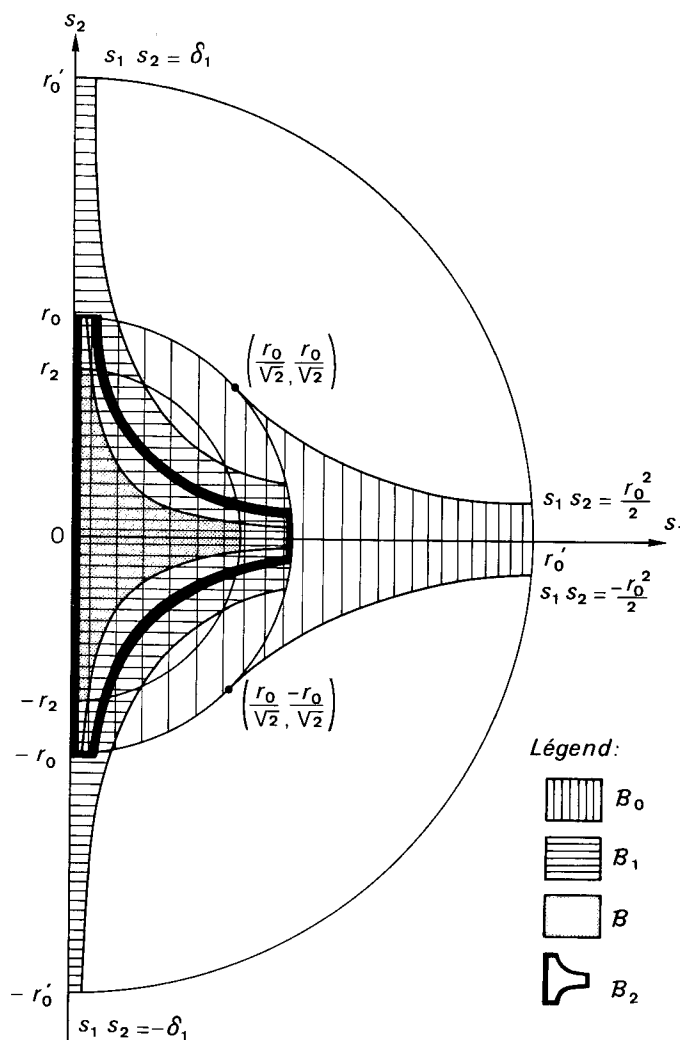


Fig. 2.

chart  $\varphi_i$ . Moreover, by (5.7),  $\gamma$  is contained in a single stable sector  $S_{ij}^s$ , the domain of  $\Phi_{ij}\varphi_i$ . We fix  $\Phi\varphi = \Phi_{ij}\varphi_i$  and  $G = G_{p(i)}$  throughout the argument. Suppose that for some  $c$ ,  $0 \leq c < k$ ,  $g^c\gamma$  lies in the domain of  $\Phi\varphi$  and  $g^c\gamma \subset B_1$ , but  $g^{c+1}\gamma \not\subset B_1$ . Since  $G(B_1) \subset \mathcal{I}_{a_i}$ ,  $g^{c+1}\gamma$  is still in the domain of  $\Phi\varphi$ . Then we have  $\Phi\varphi(g^c\gamma) \subset \mathcal{B}_1$ , but  $\Phi\varphi(g^{c+1}\gamma) = G(\Phi\varphi(g^c\gamma)) \not\subset \mathcal{B}_1$ . Thus  $\Phi\varphi(g^{c+1}\gamma)$  has a point in  $\mathcal{D}_{r_0}^c \cap \{(s_1, s_2): s_1 \geq |s_2|, |s_1 s_2| \leq \delta_1\}$  and because  $g^{c+1}\gamma \cap B \neq \emptyset$ ,  $\Phi\varphi(g^{c+1}\gamma)$  also has a point in  $\mathcal{B} \subset \mathcal{D}_{r_2}$ . But this contradicts Lemma 5.3. Therefore, for  $0 \leq c < k$ , if  $g^c\gamma$  lies in the domain of  $\Phi\varphi$  and  $g^c\gamma \subset B_1$ , then  $g^{c+1}\gamma \subset B_1$ . Since  $g^{c+1}\gamma$  is in the domain of  $\Phi\varphi$ ,  $g^{c+1}\gamma$  lies in the same component of  $B_1$  as  $g^c\gamma$ . The lemma for the case  $l=1$  follows by induction.

Now substitute  $B_0$  for  $B_1$  in the above argument. Then  $\Phi\varphi(g^{c+1}\gamma)$  would be a curve lying in  $\{(s_1, s_2): |s_1 s_2| \leq r_0^2/2, s_1 \geq 0\}$  having a point in  $\mathcal{D}_{r_2}$  and a point in  $\mathcal{D}_{r_0}^c \cap \{(s_1, s_2): s_1 \geq |s_2|\}$ . But such a curve would also have a point in  $(\mathcal{D}_{r_1} \setminus \mathcal{D}_{r_2}) \cap \{(s_1, s_2): s_1 \geq |s_2|\}$ , contradicting (3.7).  $\square$

LEMMA 5.5. — (Contraction of stable curves near singular stable leaves.)

(i) Suppose  $\gamma$  is a stable curve and  $\gamma, g\gamma, \dots, g^k\gamma$  are contained in  $B_1$ . Then:

$$l(g^k\gamma) < 2l(\gamma) + 8\varepsilon',$$

and if  $k \geq n_0$ ,  $l(g^k\gamma) < 8\varepsilon'$ .

(ii) If  $\gamma$  is a stable curve with  $l(\gamma) < \xi'$  and such that  $\gamma \cap B = \emptyset$ , while  $g\gamma, \dots, g^k\gamma$  intersect  $B$ , then:

$$l(g^k\gamma) < 2l(\gamma) + 8\varepsilon',$$

and if  $k \geq n_0$ ,  $l(g^k\gamma) < 8\varepsilon'$ .

(iii) Suppose  $\gamma$  is a stable curve and  $\gamma, g\gamma, \dots, g^k\gamma$  are contained in  $B_2$ . Then:

$$l(g^k\gamma) < 2l(\gamma) + 8\varepsilon'',$$

and if  $k \geq n_1$ ,  $l(g^k\gamma) < 8\varepsilon''$ .

(iv) If the hypothesis of (ii) holds with  $\xi'$  replaced by  $\xi$ , then:

$$l(g^k\gamma) < 2l(\gamma) + 8\varepsilon'',$$

and if  $k \geq n_1$ ,  $l(g^k\gamma) < 8\varepsilon''$ .

*Proof of (i).* — Again,  $\gamma$  is contained in a single stable sector  $S_{ij}^s$ , which is the domain of  $\Phi_{ij}\varphi_i$ . Fix  $\Phi\varphi = \Phi_{ij}\varphi_i$  and  $G = G_{p(i)}$ . Recall that the range of  $\Phi\varphi$  is  $\mathcal{D}_{r_1} \cap \{(s_1, s_2): s_1 \geq 0\}$ . Also  $G(B_1) \subset \mathcal{D}_{r_0}^c \cap \{(s_1, s_2): s_1 \geq 0\}$  and the map  $g$  is given by  $G$  (with respect to the singular chart  $\Phi\varphi$ ) in  $S_{ij}^s$ . Since  $\gamma, g\gamma, \dots, g^k\gamma$  all lie in  $B_1$ , it now follows that  $\gamma, g\gamma, \dots, g^k\gamma$  in fact lie in the same component of  $B_1$ , and therefore in the domain of  $\Phi\varphi$ .

Let  $c = (c_1, c_2)$ ,  $d = (d_1, d_2)$  be the endpoints of  $\Phi\varphi\gamma$ .

Case 1. — Assume  $k \geq n_0$ . If  $c \in \mathcal{B}_1 \cap \{(s_1, s_2): |s_2| \geq s_1\}$ , then by (5.17):

$$\bar{d}(G^{n_0}(c_1, c_2), G^{n_0}(0, c_2)) < \varepsilon',$$

and since by (5.16)  $|s_2\text{-coordinate of } G^{n_0}(0, c_2)| \leq s_2\text{-coordinate of } G^{n_0}(0, r_0') < \varepsilon'$ , it follows that  $|s_2\text{-coordinate of } G^{n_0}(c)| < 2\varepsilon'$ .

Upon iteration by  $G$ , the absolute value of the  $s_2$ -coordinate of any point decreases. Therefore:

$$|s_2\text{-coordinate of } G^k(c)| < 2\varepsilon'.$$

If  $c \in \mathcal{B}_1 \cap \{(s_1, s_2): |s_2| \leq s_1\}$ , then we already have  $|c_2| < \varepsilon'$ , because  $\delta_1 < (\varepsilon')^2$ , and consequently:

$$|s_2\text{-coordinate of } G^k(c)| < \varepsilon'.$$

Therefore in all subcases we have:

$$|s_2\text{-coordinate of } G^k(c)| < 2\varepsilon'$$

and:

$$|s_2\text{-coordinate of } G^k(d)| < 2\varepsilon'.$$

Hence the  $s_2$ -coordinates of the endpoints of  $G^k(\Phi\gamma)$  differ by less than  $4\varepsilon'$ . Thus we have:

$$l(g^k\gamma) = \bar{l}(G^k(\Phi\gamma)) < 8\varepsilon',$$

the second inequality following from the cone condition for stable curves (5.6).

Case 2. — Assume  $k < n_0$  and  $c, d \in \mathcal{B}_1 \cap \{(s_1, s_2): |s_2| \geq s_1\}$ . Again by (5.17) we have:

$$|(s_2\text{-coordinate of } G^k(c)) - (s_2\text{-coordinate of } G^k(0, c_2))| < \varepsilon'$$

and:

$$|(s_2\text{-coordinate of } G^k(d)) - (s_2\text{-coordinate of } G^k(0, d_2))| < \varepsilon'.$$

Also:

$$|(s_2\text{-coordinate of } G^k(0, c_2)) - (s_2\text{-coordinate of } G^k(0, d_2))| \leq |c_2 - d_2| \leq \bar{l}(\Phi\gamma) = l(\gamma).$$

Thus:

$$|(s_2\text{-coordinate of } G^k(c)) - (s_2\text{-coordinate of } G^k(d))| < 2\varepsilon' + l(\gamma).$$

Then applying the cone condition again, we see that:

$$l(g^k\gamma) = \bar{l}(G^k(\Phi\gamma)) < 4\varepsilon' + 2l(\gamma).$$

Case 3. — Assume  $k < n_0$  and  $c, d \in \perp_1 \setminus \{(s_1, s_2): |s_2| < s_1\}$ . Then:

$$|s_2\text{-coordinate of } G^k(c)| < \varepsilon' \text{ and } |s_2\text{-coordinate of } G^k(d)| < \varepsilon'.$$

Consequently:

$$l(g^k\gamma) < 4\varepsilon'.$$

Case 4. — Assume  $k \leq n_0$  and:

$$c \in \mathcal{B}_1 \cap \{(s_1, s_2) : |s_2| \geq s_1\}, \quad d \in \mathcal{B}_1 \cap \{(s_1, s_2) : |s_2| < s_1\}.$$

Then:

$$|(s_2\text{-coordinate of } G^k(c)) - (s_2\text{-coordinate of } G^k(0, c_2))| < \varepsilon', \quad |d_2| < \varepsilon',$$

and

$$|s_2\text{-coordinate of } G^k(d)| < \varepsilon'.$$

Also:

$$|s_2\text{-coordinate of } G^k(0, c_2)| \leq |c_2| < |c_2 - d_2| + \varepsilon' \leq \bar{l}(\Phi\varphi\gamma) + \varepsilon' = l(\gamma) + \varepsilon'.$$

Thus:

$$|s_2\text{-coordinate of } G^k(c)| < l(\gamma) + 2\varepsilon'.$$

Therefore:

$$l(g^k\gamma) \leq 2[l(\gamma) + 2\varepsilon' + \varepsilon] = 2l(\gamma) + 6\varepsilon'.$$

This completes the proof of (i).

*Proof of (ii).* — Since  $\gamma \cap B = \emptyset$  and  $g\gamma \cap B \neq \emptyset$ , there is a point  $x \in \gamma$  and a  $\Phi\varphi = \Phi_{ij}\varphi_i$  such that  $\Phi\varphi x \in \mathcal{B}_3 \cap (\mathcal{D}_{r_1} \setminus \mathcal{D}_{r_2})$ . Since  $l(\gamma) < \xi' < \bar{d}(\mathcal{B}_3 \cap \mathcal{D}_{r_1}, \mathcal{B}_1^c)$ , this implies that all of  $\gamma$  lies in the domain of  $\Phi\varphi$  and  $\Phi\varphi(\gamma) \subset \mathcal{B}_1$ . Thus  $\gamma \subset B_1$ . Now (ii) follows from (i) and Lemma 5.4.

*Proof of (iii).* — The argument is the same as for (i), because  $\mathcal{B}_2$  is defined in terms of  $\varepsilon'', n_1$  and  $r_0$  analogously to the way  $\mathcal{B}_1$  is defined in terms of  $\varepsilon', n_0$  and  $r'_0$ .

*Proof of (iv).* — The version of Lemma 5.4 with  $n=2$  and the fact that  $l(\gamma) < \xi < \bar{d}(\mathcal{B}_3 \cap \mathcal{D}_{r_1}, \mathcal{B}_2^c)$  make (iv) follow from (iii) as (ii) follows from (i).  $\square$

LEMMA 5.6. — (Controlling lengths of stable curves during one passage through  $B$ .) Suppose  $\gamma$  is a stable curve contained in  $D_{r_0}^i$  for some  $i$ ,  $\gamma, g\gamma, \dots, g^{k-1}\gamma$  intersect  $B$  and  $g^k\gamma$  does not intersect  $B$ . Then  $g^{k+a'-1}\gamma \subset D_{r_0}^c$  for some integer  $a', 0 < a' \leq a$ , where  $a$  is defined as in (5.10), and:

$$l(g^k\gamma) < J^{a'-1}l(\gamma) \leq J^{a-1}l(\gamma).$$

*Proof.* — By Lemma 5.4,  $\gamma, g\gamma, \dots, g^{k-1}\gamma$  all lie in the same component of  $B_0$ , and hence in the domain of a single  $\Phi\varphi = \Phi_{ij}\varphi_i$  such that  $\Phi\varphi(\gamma), \Phi\varphi(g\gamma), \dots, \Phi\varphi(g^{k-1}\gamma)$  all lie in  $\mathcal{B}_0$ . Moreover, since  $G(\mathcal{B}_0) \subset G(\mathcal{D}_{r_0}) \subset \mathcal{D}_{a'_1}$ ,  $g^k\gamma$  also lies in the domain of  $\Phi\varphi$ . Since  $g^{k-1}\gamma \cap B \neq \emptyset$  and  $g^k\gamma \cap B = \emptyset$ ,  $\Phi\varphi(g^{k-1}\gamma)$  has a point in  $(\mathcal{D}_{r_2} \setminus G^{-1}\mathcal{D}_{r_2})$ . Thus by (5.10) there exists an integer  $a', 0 < a' \leq a$ , such that  $G^{a'}(\Phi\varphi(g^{k-1}\gamma))$  has a point in  $(\mathcal{D}_{r_1} \setminus \mathcal{D}_{r_2}) \cap \{(s_1, s_2) : s_1 \geq |s_2|\}$ . Let  $H_1$  and  $H_2$  be branches of hyperbolas of the form  $s_1 s_2 = c_1, s_1 \geq 0, s_1 s_2 = c_2, s_1 \geq 0$  such that  $H_1$  and  $H_2$  intersect  $\Phi\varphi(\gamma)$ , and  $\Phi\varphi(\gamma)$  lies in the closed region bounded by  $H_1, H_2, \partial\mathcal{D}_{r_0}$  and  $\{(s_1, s_2) : s_1 = 0\}$ . (By property (5.8) of stable curves, these hyperbolas pass through the endpoints of  $\Phi\varphi(\gamma)$ , but we do not need this fact

here.) Then  $G^{a'}(\Phi \phi(g^{k-1}\gamma))$  also lies between  $H_1$  and  $H_2$ . From (3.7) it follows that  $G^{a'}(\Phi \phi(g^{k-1}\gamma))$  lies in  $\mathcal{D}_{r_0} \setminus \mathcal{D}_{r'_0}$  and thus  $g^{k+a'-1}\gamma \subset D_{r_0}^c$ . Also by (3.7), we have:

$$l(g^{k+a'-1}\gamma) = \bar{l}(G^{a'}(\Phi \phi(g^{k-1}\gamma))) < \bar{d}(H_1 \cap \mathcal{D}_{r_0}, H_2 \cap \mathcal{D}_{r_0}) \leq l(\gamma).$$

Thus:

$$l(g^k\gamma) \leq J^{a'-1} l(g^{k+a'-1}\gamma) < J^{a'-1} l(\gamma) \leq J^{a-1} l(\gamma). \quad \square$$

LEMMA 5.7. — (Controlling length of a stable curve at the beginning of a passage through B.) If  $\gamma$  is a stable curve with  $l(\gamma) \leq \tau$  and  $k > 0$  is such that  $g^{k-1}\gamma \cap B = \emptyset$  while  $g^k\gamma \cap B \neq \emptyset$ , then:

$$l(g^k\gamma) < C_0 J^{q+1} l(\gamma) < \tau'.$$

Proof. — Since  $g^{k-1}\gamma \cap B = \emptyset$  and  $g^k\gamma \cap B \neq \emptyset$ , there is some point  $x \in g^{k-1}\gamma$  lying in the domain of some  $\Phi \phi = \Phi_{ij}\phi_i$  such that:

$$\Phi \phi(x) \in \mathcal{B}_3 \cap (\mathcal{D}_{r_1} \setminus \mathcal{D}_{r_2}) \cap \{(s_1, s_2) : |s_2| \geq s_1 \geq 0\}.$$

Suppose that  $l(g^{k-1}\gamma) \leq \tau'$ . (This assumption will be justified below.) Then by the choice of  $\tau'$  given in (5.11) and by (5.13),  $g^{k-1}\gamma$  lies in the domain of  $\Phi \phi$  and:

$$\Phi \phi(g^{k-1}\gamma) \subset (\mathcal{D}_{r_0} \setminus \mathcal{D}_{r_3}) \cap \{(s_1, s_2) : |s_2| \geq s_1 \geq 0\}.$$

Then by (3.5), we have:

$$\begin{aligned} \Phi \phi(g^{k-1-q}\gamma) &= G^{-q}(\Phi \phi(g^{k-1}\gamma)) = F^{-q}(\Phi \phi(g^{k-1}\gamma)) \\ &\subset (\mathcal{D}_{r_0} \setminus \mathcal{D}_{r_3}) \cap \{(s_1, s_2) : |s_2| \geq s_1 \geq 0\} \\ &\subset (\mathcal{D}_{r_0} \setminus \mathcal{D}_{r_0}) \cap \{(s_1, s_2) : |s_2| \geq s_1 \geq 0\}. \end{aligned}$$

Now divide  $\gamma$  into disjoint subsets  $\gamma_0, \gamma_1, \gamma_2, \dots$ , such that  $\gamma_0 \subset D_{r_0}^c$  and for  $m > 0$  there exists a  $\overline{\Phi \phi} = \overline{\Phi_{ij}}\phi_i$  such that  $\overline{\Phi \phi}(\gamma_m) \subset \mathcal{D}_{r_0}$ ,  $\overline{\Phi \phi}(g\gamma_m) \subset \mathcal{D}_{r_2}$ ,  $\Phi \phi(g^2\gamma_m) \subset \mathcal{D}_{r_2}$ ,  $\dots$ ,  $\overline{\Phi \phi}(g^{m-1}\gamma_m) \subset \mathcal{D}_{r_2}$ ,  $\overline{\Phi \phi}(g^m\gamma_m) \subset (\mathcal{D}_{r_1} \setminus \mathcal{D}_{r_2}) \cap \{(s_1, s_2) : s_1 \geq |s_2|\}$ . (Note that since  $l(\gamma) < \tau < \tau' < r'_0 - r_0$ ,  $\gamma$  can intersect at most one component of  $D_{r_0}$  and hence only one  $\Phi_{ij}\phi_i$  is needed here.) By (3.7), we have:

$$l(g^m\gamma_m) = \bar{l}(\overline{\Phi \phi}(g^m\gamma_m)) < \bar{l}(\overline{\Phi \phi}(\gamma_m)) = l(\gamma_m).$$

For  $m > 0$  such that  $\gamma_m \neq \emptyset$ , we have  $k-1-q > m$ . Then since  $g^{k-1-q}\gamma_m$  and  $g^m\gamma_m$  both lie in  $D_{r_0}^c$ , it follows from (5.9) that:

$$l(g^{k-1-q}\gamma_m) \leq C_0 l(g^m\gamma_m) < C_0 l(\gamma_m).$$

Also, since  $\gamma_0$  and  $g^{k-1-q}\gamma_0$  both lie in  $D_{r_0}^c$ , it follows again from (5.9) that:

$$l(g^{k-1-q}\gamma_0) \leq C_0 l(\gamma_0).$$

Thus  $l(g^{k-1-q}\gamma) \leq C_0 l(\gamma)$  and consequently:

$$l(g^k \gamma) \leq C_0 J^{q+1} l(\gamma) \leq C_0 J^{q+1} \tau < \frac{\tau'}{J} < \tau'.$$

To see that  $l(g^{k-1}\gamma) < \tau'$ , we argue by contradiction and suppose that  $l(g^{k-1}\gamma) \geq \tau'$ . Then we truncate  $\gamma$  to  $\bar{\gamma}$  so that  $l(g^{k-1}\bar{\gamma}) = \tau'$  and  $g^k \bar{\gamma}$  still intersects B. Then  $l(g^k \bar{\gamma}) < \tau'/J$  which implies that  $g^{k-1}(\bar{\gamma}) < \tau'$ , a contradiction.  $\square$

*Proof of Uniform Contraction Lemma.* — Fix a stable curve  $\gamma \subset M$  with  $l(\gamma) \leq \tau$ . Fix  $n \geq \mathcal{N}$ . By (5.27), one of the following must hold:

*Case 1.* — There are at least  $n_2 + 1$  iterates among  $\gamma, g\gamma, \dots, g^n \gamma$  which lie completely in  $B^c$ .

*Case 2.* — There are at least  $\max(n_1 + 1, n_2 + 1)$  consecutive iterates among  $\gamma, g\gamma, \dots, g^n \gamma$  which lie partly in B.

*Proof for Case 1.* — Let:

$$k_0 = \min \{ j \geq 0 : g^j \gamma \subset B^c \}$$

and:

$$k_1 = \max \{ j \leq n : g^j \gamma \subset B^c \}.$$

The argument for Case 1 involves three steps: estimating, in order,  $l(g^{k_0} \gamma)$ ,  $l(g^{k_1} \gamma)$  and  $l(g^n \gamma)$ .

*Step 1.* — If  $k_0 = 0$ , then  $l(g^{k_0} \gamma) = l(\gamma) \leq \tau < \tau'$ . Suppose  $k_0 > 0$ . Then  $\gamma, g\gamma, \dots, g^{k_0-1} \gamma$  intersect B, while  $g^{k_0} \gamma$  does not intersect B. Also, since  $\gamma \cap B \neq \emptyset$  and  $l(\gamma) < \tau < \tau' < r_0 - r_2$ ,  $\gamma$  is contained in some  $D_{r_0}^i$ . Hence by Lemma 5.6,

$$l(g^{k_0} \gamma) < J^a l(\gamma) \leq J^a \tau < \tau'.$$

Thus we have  $l(g^{k_0} \gamma) < \tau'$  in both subcases.

*Step 2.* — We proceed with the estimate for  $l(g^{k_1} \gamma)$ . By the assumption of Case 1, if  $x \in \gamma$ , then  $g^{k_1} x = g_{B^c}^{j_x} g^{k_0} x$ , where  $j_x \geq n_2$ . Thus, it follows from (5.24) and (5.26) that:

$$l(g^{k_1} \gamma) \leq CK^{n_2} l(g^{k_0} \gamma) \leq CK^{n_2} \tau' < CK^{n_2} \varepsilon''.$$

*Step 3.* — Since  $\varepsilon'' < \varepsilon' < \varepsilon$ , if  $k_1 = n$ , we are done with Case 1. Suppose  $k_1 < n$ . Then  $g^{k_1} \gamma \cap B = \emptyset$ , while  $g^{k_1+1} \gamma, \dots, g^n \gamma$  intersect B. Since  $l(g^{k_1} \gamma) < \varepsilon'' < \xi'$ , by Lemma 5.5 (ii),

$$l(g^n \gamma) < 2l(g^{k_1} \gamma) + 8\varepsilon' < 2\varepsilon'' + 8\varepsilon' < 10\varepsilon' < \varepsilon.$$

*Proof for Case 2.* — Let  $k'_0 = \min \{ j \geq 0 : g^j \gamma, g^{j+1} \gamma, \dots, g^{j+\max(n_1, n_2)} \gamma \text{ all intersect B} \}$  and  $k'_1 = \max \{ k'_0 \leq j \leq n : g^{k'_0} \gamma, g^{k'_0+1} \gamma, \dots, g^j \gamma \text{ all intersect B} \}$  and let  $k_1$  be as in Case 1, if defined.

We estimate, in order,  $l(g^{k'_0} \gamma)$ ,  $l(g^{k'_1} \gamma)$ ,  $l(g^{k_1} \gamma)$  and  $l(g^n \gamma)$ .

*Step 1.* — If  $k'_0 = 0$ ,  $l(g^{k'_0}\gamma) = l(\gamma) \leq \tau < \tau'$ . Suppose  $k'_0 > 0$ . Then  $g^{k'_0-1}\gamma \subset B^c$ , while  $g^{k'_0}\gamma$  intersects  $B$ . Thus, by Lemma 5.7,

$$l(g^{k'_0}\gamma) < \tau'.$$

*Step 2.* — We now estimate  $l(g^{k'_1}\gamma')$ . Since  $g^{k'_0}\gamma$  intersects  $B \subset D_{r_2}$  and  $l(g^{k'_0}\gamma) < \tau' < r_1 - r_2$ ,  $g^{k'_0}\gamma$  lies in a component of  $D_{r_1}$  and hence in the domain of some  $\Phi \varphi = \varphi_{ij} \varphi_i$ . Divide  $\gamma$  into two pieces  $\gamma'$  and  $\gamma''$  so that  $\Phi \varphi(g^{k'_0}\gamma') \subset \mathcal{B}_3$  and  $\Phi \varphi(g^{k'_0}\gamma'') \subset \mathcal{D}_{r_1} \setminus \mathcal{B}_3$ . By the properties of stable curves given in Proposition 5.2,  $\gamma'$  is connected and  $\gamma''$  consists of at most two connected components. Note that by Lemma 5.4,  $g^{k'_0}\gamma, g^{k'_0+1}\gamma, \dots, g^{k'_1}\gamma$  all lie in the same component of  $B_0$ . Thus  $g^{k'_0}\gamma'', \dots, g^{k'_1}\gamma''$  all lie in  $B^c$ . Hence by (5.24) and (5.26), we have:

$$l(g^{k'_1}\gamma'') \leq CK^{k'_1-k'_0} l(g^{k'_0}\gamma'') \leq CK^{n_2} l(g^{k'_0}\gamma) < CK^{n_2} \tau' < CK^{n_2} < \varepsilon''.$$

Next we estimate  $l(g^{k'_1}\gamma')$ . Since  $g^{k'_0}\gamma'$  is a stable curve contained in  $B_3 \subset B_2$ , and  $g^{k'_0+1}\gamma', \dots, g^{k'_1}\gamma'$  intersect  $B$ , it follows from Lemma 5.4 that  $g^{k'_0}\gamma', \dots, g^{k'_1}\gamma'$  all lie in  $B_2$ . Then since  $k'_1 - k'_0 \geq n_1$ , Lemma 5.5 (iii) implies that:

$$l(g^{k'_1}\gamma') < 8\varepsilon''.$$

Thus  $l(g^{k'_1}\gamma) < \varepsilon'' + 8\varepsilon'' = 9\varepsilon''$ .

*Step 3.* — If  $k'_1 = n$ , we are done because  $9\varepsilon'' < 9\varepsilon' < \varepsilon$ . Thus we may assume  $k'_1 < n$ . Then  $k_1$  is defined and  $k'_1 < k_1 \leq n$ .

*Case 2.1.* —  $k_1 < n$ . Then  $g^{k_1}\gamma \cap B = \emptyset$  and  $g^{k_1+1}\gamma \cap B \neq \emptyset$ . Consequently, by Lemma 5.7, since  $l(g^{k'_1}\gamma) < 9\varepsilon'' < \tau$ ,

$$l(g^{k_1}\gamma) < C_0 J^{q+1} l(g^{k'_1}\gamma) < 9\varepsilon'' C_0 J^{q+1} < \min(\varepsilon', \xi').$$

Since  $g^{k_1}\gamma \subset B^c$  and  $g^{k_1+1}\gamma, \dots, g^n\gamma$  all intersect  $B$ , it follows from Lemma 5.5 (ii) that:

$$l(g^n\gamma) < 2l(g^{k_1}\gamma) + 8\varepsilon' < 2\varepsilon' + 8\varepsilon' = 10\varepsilon' < \varepsilon.$$

*Case 2.2.* —  $k_1 = n$  and  $g^{k_1+1}\gamma \cap B \neq \emptyset$ . Then the first estimate in the argument for Case 2.1 gives  $l(g^n\gamma) < \varepsilon' < \varepsilon$ .

*Case 2.3.* —  $k_1 = n$  and  $g^{k_1+1}\gamma \subset B^c$ . Assume that  $l(g^n\gamma) \leq \tau' < r'_0 - r_0$ . (This assumption will be justified later.) Then  $g^n\gamma$  intersects at most one component of  $D_{r_0}$ . Now divide  $\gamma$  into pieces  $\gamma_1, \gamma_2, \gamma_3$  such that  $g^n\gamma_1 \subset (B_1 \cap D_{r_0}) \setminus B$ ,  $g^n\gamma_2 \subset D_{r_0} \setminus B_1$ , and  $g^n\gamma_3 \subset D_{r_0}^c$ . Then by property (5.8) of stable curves,  $\gamma_1$  is connected, but  $\gamma_2$  and  $\gamma_3$  need not be. Note that by Lemma 5.6, since  $g^{k'_0}\gamma \subset D_{r_0}$  and  $g^{k'_0}\gamma, \dots, g^{k'_1}\gamma$  intersect  $B$ , while  $g^{k'_1+1}\gamma \subset B^c$ ,

$$(5.28) \quad g^{k'_1+a'}\gamma \subset D_{r_0}^c, \text{ for some } a', 0 < a' \leq a.$$

Thus, by (5.9), since  $g^n\gamma_3 \subset D_{r_0}^c$ , we have:

$$l(g^n\gamma_3) \leq J^{a'} C_0 l(g^{k'_1}\gamma_3) \leq J^a C_0 l(g^{k'_1}\gamma_3).$$



Also, since by (5.18) it is possible to move any point on  $g^n \gamma_2$  out of  $D_{r_0}$  in at most  $b$  iterates of  $g^{-1}$ , (5.9) and (5.28) imply that:

$$l(g^n \gamma_2) \leq J^{a+b} C_0 l(g^{k_1} \gamma_2).$$

Finally we estimate  $l(g^n \gamma_1)$ . Let  $i$  be the first positive integer such that for some point  $c \in \Phi \phi(g^n \gamma_1)$ ,  $G^{-i} c \in \mathcal{D}_{r'_1} \setminus \mathcal{D}_{r'_2}$ . Then  $g^{n-i} \gamma_1, \dots, g^n \gamma_1$  all lie in  $B_1$ . Now if  $l(g^{n-i} \gamma_1) \leq \min(r'_0 - r'_1, r'_2 - r'_0)$ , we have:

$$\Phi \phi(g^{n-i} \gamma_1) \subset (\mathcal{D}_{r'_0} \setminus \mathcal{D}_{r'_2}) \cap \mathcal{B}_1.$$

Since  $g^{n-i} \gamma_1 \subset D_{r'_0}^c$  and  $g^{k'_1+a'} \gamma_1 \subset D_{r'_0}^c$ , where  $0 < a' \leq a$ , and  $k'_1 < n-i$ ,

$$l(g^{n-i} \gamma_1) \leq J^a C_0 l(g^{k'_1} \gamma_1) < J^a C_0 (9 \varepsilon'') < \tau' < \min(r'_0 - r'_1, r'_2 - r'_0).$$

Thus we can use a truncation argument, as in the proof of Lemma 5.7, to see that, in fact, we must have  $l(g^{n-i} \gamma_1) \leq \min(r'_0 - r'_1, r'_2 - r'_0)$  to begin with and, consequently,  $l(g^{n-i} \gamma_1) \leq 9 J^a C_0 \varepsilon''$ . Since  $g^{n-i} \gamma_1, \dots, g^n \gamma_1$  all lie in  $B_1$ . It follows from Lemma 5.5 (i),

$$l(g^n \gamma_1) < 2 l(g^{n-i} \gamma_1) + 8 \varepsilon' < 18 J^a C_0 \varepsilon'' + 8 \varepsilon'.$$

Hence:

$$\begin{aligned} l(g^n \gamma) &= l(g^n \gamma_1) + l(g^n \gamma_2) + l(g^n \gamma_3) \\ &< 18 J^a C_0 \varepsilon'' + 8 \varepsilon' + J^{a+b} C_0 l(g^{k_1} \gamma_2) + J^{a+1} C_0 l(g^{k_1} \gamma_3) \\ &< 18 J^a C_0 \varepsilon'' + 8 \varepsilon' + J^{a+b} C_0 (9 \varepsilon'') \\ &< 27 C_0 J^{a+b} \varepsilon'' + 8 \varepsilon' < 9 \varepsilon' < \min(\varepsilon, \tau'). \end{aligned}$$

Now another truncation argument can be used to justify the initial assumption that  $l(g^n \gamma) \leq \tau'$ . Therefore  $l(g^n \gamma) < \varepsilon$ .  $\square$

## 6. Corollaries of Uniform Contraction Lemma and Uniqueness of Stable Curves

**COROLLARY 6.1** (Global contraction). — *For every stable curve  $\gamma$ , the length  $l(g^n \gamma) \rightarrow 0$  as  $n \rightarrow \infty$  and this convergence is uniform for all curves of uniformly bounded length.*

*Proof.* — Let  $K(\gamma) = [l(\gamma)/\tau] + 1$ . Let us divide  $\gamma$  into pieces  $\gamma_1, \dots, \gamma_{K(\gamma)}$  such that  $l(\gamma_i) < \tau$ ,  $i=1, \dots, K(\gamma)$ . By the Uniform Contraction Lemma, we have  $l(g^n \gamma_i) \rightarrow 0$  uniformly for all  $\gamma$  and all  $i=1, \dots, K(\gamma)$ .  $\square$

**COROLLARY 6.2** (Uniform expansion). — *For any  $\varepsilon > 0$ ,  $R > 0$ , there exists a natural number  $\mathcal{N}(\varepsilon, R)$  such that for every stable curve  $\gamma$  with  $l(\gamma) \geq \varepsilon$  and for every  $n \geq \mathcal{N}(\varepsilon, R)$ , we have  $l(g^{-n} \gamma) > R$ .*

*Proof.* — Take  $\mathcal{N}(\varepsilon, R) = \mathcal{N}(\varepsilon/([R/\tau] + 1))$ , where  $\mathcal{N}(\delta)$  is the function from the Uniform Contraction Lemma, and divide  $\gamma$  into  $[R/\tau] + 1$  pieces of equal length. The length of every

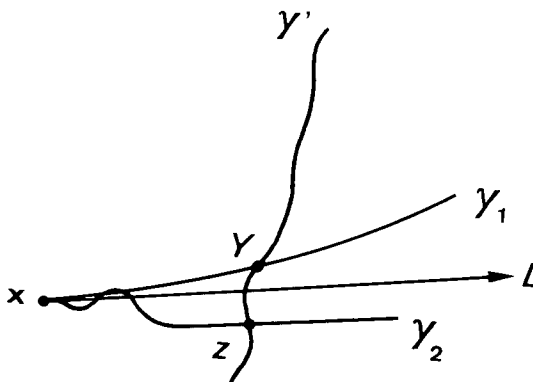


Fig. 3

piece is  $\geq \varepsilon/[\mathbf{R}/\tau] + 1$ . Consequently, by the Uniform Contraction Lemma, the length of the  $n$ -th pre-image of every piece is  $\geq \tau$  and  $l(g^{-n}\gamma) \geq \tau([\mathbf{R}/\tau] + 1) > \mathbf{R}$ .  $\square$

**COROLLARY 6.3** (Expansion and contraction for unstable curves). — *For every unstable curve  $\gamma$ ,  $l(g^{-n}\gamma) \rightarrow 0$  as  $n \rightarrow \infty$ , the convergence being uniform for curves of uniformly bounded length. For every  $\varepsilon > 0$ ,  $\mathbf{R} > 0$ , there exists a natural number  $\mathcal{N}_1(\varepsilon, \mathbf{R})$  such that for an unstable curve  $\gamma$  with  $l(\gamma) \geq \varepsilon$  and for  $n \geq \mathcal{N}_1(\varepsilon, \mathbf{R})$ , we have  $l(g^n\gamma) > \mathbf{R}$ .*

*Proof.* — Apply Corollaries 6.1 and 6.2 to the map  $g^{-1}$ . Unstable curves for  $g$  are stable curves for  $g^{-1}$ .  $\square$

**PROPOSITION 6.1** (Uniqueness of stable curves). — *Let  $x \in \mathbf{M} \setminus \{x_1, \dots, x_m\}$ , and let  $L$  be one of two rays of the space  $E_{x,g}^s$ . Let  $\gamma_1, \gamma_2$  be two stable curves which begin at  $x$  and are tangent to  $L$ . Then one of the curves is a part of the other.*

*Proof.* — Suppose that the statement were not correct. Then, replacing, if necessary,  $x$  by another point on  $\gamma_1 \cap \gamma_2$ , we can keep all the assumptions and also assume that arbitrarily close to  $x$  there are both points belonging to  $\gamma_1 \setminus \gamma_2$  and to  $\gamma_2 \setminus \gamma_1$ .

Let us choose a point  $y \in \gamma_1 \setminus \gamma_2$ . Since the angle between  $E_{x,g}^s$  and  $E_{x,g}^u$  is bounded away from 0 outside any fixed neighborhood of  $\{x_1, \dots, x_m\}$ , if  $y$  is chosen sufficiently close to  $x$ , then there is a short unstable curve  $\gamma'$  which passes through  $y$  and intersects  $\gamma_2$ . (Recall that close to  $x$  the tangents to  $\gamma_1$  and  $\gamma_2$  are almost parallel.) Let us denote the point of intersection of  $\gamma_2$  and  $\gamma'$  by  $z$  (cf. Fig. 3). Obviously,

$$(6.1) \quad d(g^n(y), g^n(z)) \leq d(g^n(y), g^n(x)) + d(g^n(z), g^n(x)) \leq l(g^n\gamma_1) + l(g^n\gamma_2).$$

(Recall that  $d$  is the distance generated by our metric  $\rho$  (cf. paragraph 3).)

Thus, by Corollary 6.1,  $d(g^n(y), g^n(z))$  goes to 0 as  $n \rightarrow \infty$  uniformly for all  $y$  sufficiently close to  $x$ .

On the other hand, by Corollary 6.3, the length of the images of the segment of  $\gamma'$  which connects  $y$  and  $z$  goes to  $\infty$  as  $n \rightarrow \infty$ . Furthermore, there exist constants  $\varepsilon_0 > 0$ ,  $C > 0$ , such

that if an unstable curve  $\Gamma$  of length  $\leq \varepsilon_0$  connects the points  $y_1, y_2$  then:

$$(6.2) \quad d(y_1, y_2) > C l(\Gamma).$$

Choosing the point  $y$  sufficiently close to  $x$ , one can assure that for some  $n > \mathcal{N}(C\varepsilon_0/2J)$ ,

$$(6.3) \quad J^{-1}\varepsilon_0 \leq l(g^n \gamma') \leq \varepsilon_0,$$

where  $J = \max_x \|dg_x\|$ .

It follows from (6.2) and (6.3) that:

$$d(g^n y, g^n z) > J^{-1} C \varepsilon_0,$$

which by (6.1) contradicts the Uniform Contraction Lemma.  $\square$

### 7. Topological Conjugacy of the Pseudo-Anosov Map $f$ and the Diffeomorphism $g$

Let us now consider one of the Markov rectangles  $R_i$  provided with a coordinate system  $(u, v)$  determined by the invariant foliations and invariant measures of the map  $f$  (cf. paragraph 2); for the sake of brevity, we will omit the index showing the dependence of the coordinate system on  $i$ . The "vertical" sides of the rectangle  $u=0$  and  $u=l_i$  are pieces of stable leaves of some of the singular points; they will remain stable curves for  $g$  as well (cf. paragraph 5). Furthermore, a connected component of the intersection of a (sufficiently large) stable curve with  $R_i$  has the form:

$$(7.1) \quad \text{graph } \varphi = \{(\varphi(v), v), \text{ where } |\varphi(v_1) - \varphi(v_2)| < |v_1 - v_2|\}.$$

Obviously, every connected stable curve in  $R_i$  can be extended to a curve of the form (7.1). Thus we can call every stable curve of the form (7.1) a *maximal stable curve* in  $R_i$ .

Let us recall that we can associate with  $f$  the following  $N \times N$  (0-1) matrix:

$$A^f = (a_{ij}), \quad a_{ij} = \begin{cases} 1 & \text{if } \text{Int}(f R_i \cap R_j) = \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

By (3.3),  $g R_i = f R_i$ , for every  $i = 1, \dots, N$ . Thus the corresponding matrix  $A^g$  coincides with  $A^f$ . Henceforth, we will denote this matrix simply by  $A$ .

As in paragraph 2.4, a finite sequence  $\sigma^{(n)} = (\sigma_0, \dots, \sigma_{n-1})$ ,  $\sigma_i \in \{1, \dots, N\}$ , is called *admissible* if  $a_{\sigma_{i-1}\sigma_i} = 1$  for  $i = 1, \dots, n-1$ . Accordingly an infinite sequence  $\sigma = (\sigma_0, \sigma_1, \dots)$  is *admissible* if  $a_{\sigma_{i-1}\sigma_i} = 1$  for  $i = 1, 2, \dots$ . Let us denote for any admissible sequence  $\sigma^{(n)} = (\sigma_0, \dots, \sigma_{n-1})$ , the following set:

$$R_{\sigma^{(n)}}^g = \bigcap_{i=0}^{n-1} g^{-i} R_{\sigma_i}.$$

Since the pre-image of a stable curve is a stable curve and the partition  $\{R_1, \dots, R_N\}$  has the Markov property with respect to  $g$ , one can see that in the Markov coordinates  $(u, v)$  in  $R_{\sigma_0}$ , the set  $R_{\sigma^{(n)}}^g$  has the form:

$$(7.2) \quad \{(u, v) : \varphi_1(v) \leq u \leq \varphi_2(v)\},$$

where graph  $\varphi_1$  and graph  $\varphi_2$  are two stable curves, and consequently  $\varphi_1(v) < \varphi_2(v)$  and:

$$(7.3) \quad \begin{cases} |\varphi_1(v_1) - \varphi_1(v_2)| \leq |v_1 - v_2|, \\ |\varphi_2(v_1) - \varphi_2(v_2)| \leq |v_1 - v_2|. \end{cases}$$

PROPOSITION 7.1. — *For any admissible infinite sequence  $\sigma = (\sigma_0, \sigma_1, \dots)$  the set:*

$$R_\sigma^g = \bigcap_{i=0}^{\infty} g^{-i} R_{\sigma_i},$$

*is a maximal stable curve in  $R_{\sigma_0}$ .*

*Proof.* — Obviously,  $R_\sigma^g = \bigcap_{i=0}^{\infty} R_{\sigma^{(n)}}^g$ , where  $\sigma^{(n)} = (\sigma_0, \dots, \sigma_{n-1})$ . Consequently,  $R_\sigma^g$  has the form (7.2), where (7.3) is satisfied. If  $\varphi_1 \equiv \varphi_2$ , then the maximal stable curve in  $R_{\sigma_0}$  which begins at the point  $(\varphi_1(0), 0)$  must coincide with  $R_\sigma^g$ . For, Proposition 6.1 assures that this maximal stable curve must lie inside every set  $R_{\sigma^{(n)}}^g$  and  $R_\sigma^g$  is the only set of the form (7.2) which contains the point  $(\varphi_1(0), 0)$  and satisfies that condition. So it remains to prove that  $\varphi_1 \equiv \varphi_2$ .

Let  $R_{\sigma^{(n)}}^g = \{(u, v) : \varphi_1^{\sigma^{(n)}}(v) \leq u \leq \varphi_2^{\sigma^{(n)}}(v)\}$ . Then the sequence of functions  $\varphi_1^{\sigma^{(n)}}$  is non-decreasing, and converges pointwise to  $\varphi_1$  and similarly for  $\varphi_2^{\sigma^{(n)}}$  and  $\varphi_2$ . The Lipschitz condition (7.3) is satisfied for all  $\varphi_1^{\sigma^{(n)}}$  and  $\varphi_2^{\sigma^{(n)}}$ . Consequently, the convergence of  $\varphi_1^{\sigma^{(n)}}$  to  $\varphi_1$  and of  $\varphi_2^{\sigma^{(n)}}$  to  $\varphi_2$  is uniform.

On the other hand, let us consider the set  $g^{n-1} R_{\sigma^{(n)}}^g \subset R_{\sigma_{n-1}}$ . The images of the curves graph  $\varphi_1^{\sigma^{(n)}}$  and graph  $\varphi_2^{\sigma^{(n)}}$  are pieces of two components of the stable boundary of the rectangle  $R_{\sigma_{n-1}}$ . Every maximal unstable curve in  $R_{\sigma_{n-1}}$  which begins at a point  $y \in g^{n-1}(\text{graph } \varphi_1^{\sigma^{(n)}})$  stays within the set  $g^{n-1} R_{\sigma^{(n)}}^g$  (uniqueness of unstable curves). Hence, we can connect every such point  $y$  to a point at  $g^{n-1}(\text{graph } \varphi_2^{\sigma^{(n)}})$  by an unstable curve. Applying Corollary 6.3, we can see that given  $\varepsilon > 0$ , there exists  $\mathcal{N}_1(\varepsilon)$  such that for  $n > \mathcal{N}_1(\varepsilon)$ , the distance from every point  $x \in \text{graph } \varphi_1^{\sigma^{(n)}}$  to graph  $\varphi_2^{\sigma^{(n)}}$  is less than  $\varepsilon$ . Letting  $n \rightarrow \infty$  and using the uniform convergence of  $\varphi_i^{\sigma^{(n)}}$  to  $\varphi_i$ ,  $i = 1, 2$ , we conclude that graph  $\varphi_1$  belongs to the closure of graph  $\varphi_2$ , i. e.  $\varphi_1 \equiv \varphi_2$ .  $\square$

The maximal stable curve in  $R_i$  which begins at the point  $(u, 0)$  can be represented in the form graph  $\varphi_u$ .

PROPOSITION 7.2. — *The function  $\varphi(u, v)$  defined by  $\varphi(u, v) = \varphi_u(v)$  is continuous.*

*Proof.* — Let us fix  $u_0$  and represent the maximal stable curve graph  $\varphi_{u_0}$  as  $R_\sigma^g$  for some admissible infinite sequence  $\sigma = (i, \sigma_1, \sigma_2, \dots)$ . If this representation is unique, then for every  $n$ ,  $\varphi_1^{\sigma^{(n)}} < \varphi_{u_0} < \varphi_2^{\sigma^{(n)}}$ , the functions  $\varphi_1^{\sigma^{(n)}}$  converge uniformly to  $\varphi_{u_0}$  from below, and the

$\varphi_2^{\sigma(n)}$  converge uniformly to  $\varphi_{u_0}$  from above. If  $\varphi_1^{\sigma(n)}(0) < u < \varphi_2^{\sigma(n)}(0)$ , then  $\varphi_1^{\sigma(n)}(v) < \varphi_u(v) < \varphi_2^{\sigma(n)}(v)$ . Thus, the family  $\varphi_u$  is continuous at  $u_0$ .

If the representation is not unique, then for some  $n_0$  either  $\varphi_{u_0} = \varphi_1^{\sigma(n_0)}(v)$  or  $\varphi_{u_0} = \varphi_2^{\sigma(n_0)}(v)$ . In the first case there is exactly one more admissible sequence  $\tilde{\sigma} = (i, \tilde{\sigma}_1, \dots)$  such that  $\varphi_{u_0} = \varphi_2^{\tilde{\sigma}(n_0)}$ . Thus, for all  $n$ ,  $\varphi_1^{\tilde{\sigma}(n)} < \varphi_{u_0} < \varphi_2^{\sigma(n)}$  and both sequences  $\varphi_1^{\tilde{\sigma}(n)}$  and  $\varphi_2^{\sigma(n)}$  converge to  $\varphi_{u_0}$ . Now the same arguments as above work. The second case is similar.  $\square$

*Remarks 1.* — Since the line field  $E_g^s$  is continuous outside the singular points, it follows from Proposition 7.2 that for  $u \neq 0$ , the family of functions  $\varphi_u$  is continuous in  $C^1$  topology.

2. Since the Markov coordinate systems in adjacent rectangles agree, Proposition 7.2 implies that stable curves form a continuous foliation with singularities at the points  $x_1, \dots, x_m$ . The same arguments show that unstable curves form another continuous foliation transversal to the first one outside the singularities. Let us denote these stable and unstable foliations by  $\mathfrak{F}_g^s$  and  $\mathfrak{F}_g^u$ , respectively.

Now we are prepared to carry out the final step in the proof of topological conjugacy of the pseudo-Anosov map  $f$  and the diffeomorphism  $g$ . Let us recall several notations. The eigenvalue of the matrix  $A$  with maximal absolute eigenvalue is denoted by  $\lambda$ . Actually  $\lambda$  is positive, and  $\log \lambda$  is equal to the topological entropy of the map  $f$ . A positive eigenvector of  $A$  corresponds to the eigenvalue  $\lambda$ . We denote this vector  $l = (l_1, \dots, l_N)$ . We assume that the vector  $l$  is normalized, i. e.,  $\sum_{i=1}^N l_i = 1$ . By adjusting the transverse measure  $\mu^s = \mu_f^s$  by a constant multiple if necessary, we may also assume that  $l_i$  is equal to the transverse measure  $\mu_f^s(\gamma_u^i)$ , where  $\gamma_u^i$  is an unstable side of the rectangle  $R_i$ . Similarly, if we let  $\bar{l} = (\bar{l}_1, \dots, \bar{l}_N)$  be the normalized positive eigenvector of the adjoint matrix  $A^*$ , then  $\bar{l}_i$  may be assumed to be equal to  $\mu_f^u(\gamma_s^i)$ .

**PROPOSITION 7.3.** — *There exist non-atomic transversal invariant measures  $\mu_g^s$  and  $\mu_g^u$  for the foliations  $\mathfrak{F}_g^s$  and  $\mathfrak{F}_g^u$  such that:*

$$(i) \quad g_* \mu_g^s = \lambda^{-1} \mu_g^s \quad \text{and} \quad g_* \mu_g^u = \lambda \mu_g^u,$$

$$(ii) \quad \mu_g^s(\gamma_u^i) = l_i \quad \text{and} \quad \mu_g^u(\gamma_s^i) = \bar{l}_i, \quad i = 1, \dots, N,$$

(iii)  $\mu_g^s$  and  $\mu_g^u$  are positive on every open transversal interval.

*Proof.* — We will give the proof only for the stable foliation. The unstable foliation is considered similarly.

Since the two foliations are transversal, in order to define the measure  $\mu_g^s$ , it suffices to define it for unstable curves. Moreover, we can do it separately in each rectangle  $R_i$  and only for arcs on the unstable boundary of the rectangle, and then check that the measures agree on the intersection of any two rectangles.

We begin with the definition of  $\mu_g^s$  on  $\gamma_u^i$  by (ii). Since:

$$(7.4) \quad \sum_{j=1}^N a_{ij} l_j = \lambda l_i,$$

condition (i) is satisfied for maximal unstable curves in  $R_i$ .

For every  $n \geq 1$ , we have:

$$R_i = \bigcup_{\sigma^{(n)}} R_{\sigma^{(n)}},$$

where the union is taken over the set of all admissible sequences  $\sigma^{(n)} = (i, \sigma_1, \dots, \sigma_{n-1})$ . Let  $\gamma_{\sigma^{(n)}}$  be an intersection of a component of the unstable boundary  $R_i$  with  $R_{\sigma^{(n)}}$ . We define:

$$(7.5) \quad \mu_g^s(\gamma_{\sigma^{(n)}}) = \lambda^{-n} l_{\sigma_{n-1}}.$$

It follows easily from (7.4) that this definition is consistent, i.e., for every admissible sequence  $\sigma^{(m)} = (i, \sigma_1, \dots, \sigma_{m-1})$  and every  $n > m$ , we have:

$$\mu_g^s(\gamma_{\sigma^{(m)}}) = \sum_{\sigma^{(n)} = (\sigma^{(m)}, \sigma^{(n-m)})} \mu_g^s(\gamma_{\sigma^{(n)}}),$$

where the summation is taken over all admissible sequences  $\sigma^{(n)}$  which begin with  $\sigma^{(m)}$ .

It follows from Corollary 6.3 that the maximal length of  $\gamma_{\sigma^{(n)}}$  over all admissible  $\sigma^{(n)}$  goes to 0 as  $n \rightarrow \infty$ . Thus, applying a standard approximation procedure, we can define  $\mu_g^s(\alpha)$  for every arc  $\alpha$  of the unstable boundary of  $R_i$ .

However, the same arc  $\alpha$  has an intersection with a boundary component of at least one more rectangle. In fact, cutting  $\alpha$  into pieces, we can consider each piece separately and assume that from the beginning  $\alpha$  belongs to exactly one more rectangle. Thus  $\alpha \subset \gamma \cap \gamma'$ , where  $\gamma$  and  $\gamma'$  are components of the unstable boundaries of the rectangles  $R_i$  and  $R_j$ , respectively. Let us temporarily denote the measure  $\mu_g^s$  defined in  $R_i$  and  $R_j$  by  $\mu_i$  and  $\mu_j$ , respectively. So we have to show that:

$$(7.6) \quad \mu_i(\alpha) = \mu_j(\alpha).$$

DEFINITION. — We will call the curve  $\gamma_{\sigma^{(n)}}$  a *regular component of  $\gamma$*  if  $g^n \gamma_{\sigma^{(n)}} \subset \text{Int } R_{\sigma_n}$ .

If  $\gamma_{\sigma^{(n)}}$  is a regular component of  $\gamma$ , then it is also a regular component of  $\gamma'$ , and by (7.5), obviously:

$$(7.7) \quad \mu_i(\gamma_{\sigma^{(n)}}) = \mu_j(\gamma_{\sigma^{(n)}}).$$

For  $\alpha \subset \gamma$ , let us denote by  $\underline{\alpha}_n$  the union of all  $\gamma_{\sigma^{(n)}}$  which lie inside  $\alpha$  and by  $\bar{\alpha}_n$  the union of all  $\gamma_{\sigma^{(n)}}$  which have a non-empty intersection with  $\alpha$ . Obviously,

$$\mu_i(\alpha) = \lim_n \mu_i(\underline{\alpha}_n) = \lim_n \mu_i(\bar{\alpha}_n).$$

Equality (7.6) follows immediately from (7.7) and the following lemma.

LEMMA. — Let  $\alpha \subset \gamma$  and suppose both ends of  $\alpha$  are different from singular points  $x_1, \dots, x_m$ . Then for any sufficiently large  $n$  all  $\gamma_{\sigma^{(n)}}$  which belong to  $\bar{\alpha}_n$  are regular components of  $\gamma$ .

*Proof of the Lemma.* — The unstable boundary of the partition  $\{R_1, \dots, R_N\}$  is a union of arcs  $\Gamma_1, \dots, \Gamma_l$  of unstable manifolds of singular points  $x_1, \dots, x_m$ . (This is true for both maps  $f$  and  $g$ .) In particular, let  $x \in \Gamma_r$  be a point which lies between the singularity and  $\alpha$ . Let us denote the arc of  $\Gamma_r$  from the singularity to  $x$  by  $\Gamma$  and the rest of  $\Gamma_r$  by  $\Gamma'$ . By Corollary 6.3, there exists  $N$  which depends only on the length of  $\Gamma$  such that for  $n \geq N$ , the length  $l(g^n \Gamma) > l(\Gamma_r)$ . This means that  $g^n \Gamma$  contains  $\Gamma_r$  and consequently  $g^n \Gamma'$  is disjoint from the unstable boundary of the partition  $\{R_1, \dots, R_N\}$ . If  $n$  is large enough  $\gamma_{\sigma^{(n)}} \subset \bar{\alpha}_n$  implies that  $\gamma_{\sigma^{(n)}} \subset \Gamma'$ . Thus  $\gamma_{\sigma^{(n)}}$  is a regular component of  $\gamma$ .

To finish the proof of Proposition 7.3, it is enough to notice that every unstable curve can be divided into connected pieces belonging to rectangles  $R_i$ , and within any rectangle, for every unstable curve  $\alpha$ , one can find a curve  $\tilde{\alpha}$  on the unstable boundary whose ends belong to the same stable curves. We let by definition  $\mu_s^g(\alpha) = \mu_i(\tilde{\alpha})$ . Equalities (7.5) and (7.6) show that the definition is consistent and (i) holds; (iii) follows from Corollary 6.1.  $\square$

The measures  $\mu_s^g$  and  $\mu_u^g$  provide a local coordinate system in every rectangle  $R_i$ . Namely, let us fix a corner  $z_i$  in every rectangle. Then for  $x \in R_i$  we can draw maximal stable and unstable curves in  $R_i$  which intersect the unstable and stable components of  $\partial R_i$  containing  $z_i$  at the points  $x'$  and  $x''$ , respectively. Then we let:

$$(6.8) \quad \begin{cases} u(x) = \mu_u^g((x'', x)), \\ v(x) = \mu_s^g((x', x)). \end{cases}$$

These coordinates change by a translation when we pass from  $R_i$  to  $R_j$  along a boundary component.

**THEOREM.** — *The diffeomorphism  $g$  is topologically conjugate with the pseudo-Anosov map  $f$ , and the conjugating homeomorphism is isotopic to the identity.*

*Proof.* — The homeomorphism  $h$  conjugating  $g$  and  $f$  is defined in the following way:

- (i)  $h$  is identical at all vertices of all  $R_i$ 's;
- (ii)  $h(R_i) = R_i$ ;
- (iii) within  $R_i$ ,  $h(x)$  is defined by the points whose Markov coordinates are equal to coordinates of  $x$  defined by (6.8). It follows immediately from Proposition 7.2 that  $h$  is a homeomorphism of  $M$  and  $g = h^{-1}fh$ . Also (i) and (ii) imply that  $h$  is isotopic to the identity.  $\square$

The fact that  $g$  is Bernoulli with respect to a smooth measure can be derived from results in [P], in our case exactly as in [K, paragraph 4].

## 8. Generalizations

The results of this paper can be generalized in several directions:

1. Diffeomorphisms with stable and unstable measured foliations having one-prong singularities. A one-prong singularity corresponds to a local chart satisfying (2.1) with  $p=1$ . Bernoulli diffeomorphisms on the sphere constructed in [K] have four such

singularities (and no multi-prong singularities). In [F-L-P], maps having this type of singularity as well as those previously allowed are called generalized pseudo-Anosov maps. Any generalized pseudo-Anosov map can be modified by a local perturbation consisting of a "slowing-down" and "blowing-up" procedure described in [K, paragraph 2] to produce a diffeomorphism preserving a smooth measure which is topologically conjugate to the original map *via* a homeomorphism isotopic to the identity. Here the "blowing-up" part of the construction cannot be omitted as was possible in the previous case, but with this additional step the arguments given above still work.

2. Pseudo-Anosov maps on surfaces with boundary. Singularities on the boundary correspond to local charts satisfying (2.1) with  $D_{a_i}$  replaced by  $D_{a_i} \cap \{z : \operatorname{Re} z \geq 0\}$  or by  $D_{a_i} \cap \{z : \operatorname{Re} z \leq 0\}$ . The theory of such maps is outlined in [F-L-P, 11].

The referee pointed out that their description contains a mistake noticed by Jiang. Namely, a Pseudo-Anosov map should not be required to be the identity of the boundary. For example, a boundary component may consist of two hyperbolic fixed points  $x$  and  $y$  together with two branches of the stable manifold of  $x$  which at the same time is the unstable manifold of  $y$ . Our smoothing construction can be applied to this more general case.

3. Similar comments can be made for pseudo-Anosov maps on non-orientable surfaces with or without boundary. As soon as the generalization of the topological theory mentioned in paragraph 1 of [F-L-P] is developed, the smoothing procedure is completely parallel to that in the orientable case.

Of course, the complications and generalizations given above may be combined and dealt with in the same manner.

#### REFERENCES

- [B-F-K] M. I. BRIN, J. FELDMAN and A. KATOK, *Bernoulli Diffeomorphisms and Group Extensions of Dynamical Systems with Non-Zero Characteristic Exponents* (*Annals of Math.*, Vol. 113, 1981, pp. 159-179).
- [F-L-P] A. FATHI, F. LAUDENBACH and V. POËNARU, *Asterisque*, Vol. 66-67, *Travaux de Thurston sur les surfaces* (Séminaire Orsay).
- [F] J. FRANKS, *Anosov Diffeomorphisms* (*Proceedings of a Symposium in Pure Mathematics*, Vol. 14, *Global Analysis*, A.M.S., 1970, pp. 61-93).
- [G] A. A. GURA, *On Expansive Diffeomorphisms of the Torus* (*Math. notes*, Vol. 18, No. 1, 1975, pp. 605-610. Translation from Russian).
- [K] A. KATOK, *Bernoulli Diffeomorphisms on Surfaces* (*Annals of Math.*, Vol. 110, 1979, pp. 529-547).
- [O-R] T. O'BRIEN and W. REDDY, *Each Compact Orientable Surface of Positive Genus Admits an Expansive Homeomorphism* (*Pacific J. of Math.*, Vol. 35, No. 3, 1970, pp. 737-741).
- [O] D. ORNSTEIN, *Ergodic Theory, Randomness and Dynamical Systems*, Yale University Press, 1974.
- [P] Ya. B. PESIN, *Characteristic Lyapunov Exponents and Smooth Ergodic Theory* (*Russian Math. Surveys*, Vol. 32, No. 4, 1977, pp. 55-114).
- [S] S. SMALE, *Differentiable Dynamical Systems* (*Bull. A.M.S.*, Vol. 73, 1967, pp. 747-817).
- [T] W. THURSTON, *On the Geometry and Dynamics of Diffeomorphisms of Surfaces*, preprint.

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