Local differentiable rigidity of some partially hyperbolic actions of higher rank abelian groups

Anatole Katok

(Penn State University)

joint work with

Danijela Damjanović

(EPDI–IHES)

Actions of higher rank abelian groups: $\mathbb{Z}^k \times \mathbb{R}^l,\ k+l \geq 2$

PROGRAM: Establish local differentiable rigidity for broad classes of partially hyperbolic algebraic actions of $\mathbb{Z}^k \times \mathbb{R}^l,\ k+l \geq 2$, i.e. homogeneous or affine actions on homogeneous and double coset spaces for Lie groups.

The most general condition leading to rigidity phenomena (local rigidity, measure rigidity, cocycle rigidity, ...):

($\mathfrak{R}$) *The group $\mathbb{Z}^k \times \mathbb{R}^l$ contains a subgroup L isomorphic to $\mathbb{Z}^2$ such that for the suspension of the restriction of the action to L every element other than identity acts ergodically with respect to the standard invariant measure obtained from Haar measure.*

For an action of $\mathbb{Z}^k$, $(k \geq 2)$ by automorphisms of the torus $(\mathfrak{R})$ is equivalent to

- Existence of a subgroup isomorphic to $\mathbb{Z}^2$ which acts (with the exception of identity) by ergodic automorphisms, or
- Absence of non-trivial rank one factors.

Local differentiable rigidity is proved in

D. Damjanovic and **A. Katok**, Local Rigidity of Partially Hyperbolic Actions I. KAM method and actions on the Torus, www.math.psu.edu/katok_a/papers.html

Weyl chamber flow on $SL(n, \mathbb{R})/\Gamma,\ n \geq 3$

We consider another representative case of actions satisfying $(\mathfrak{R})$: generic restrictions of Weyl chamber flows.
We prove local differentiable rigidity by
A METHOD TOTALLY DIFFERENT FROM THE METHOD used for actions on the torus.

Let $X \stackrel{\text{def}}{=} SL(n, \mathbb{R})/\Gamma,\ n \geq 3$ where Γ is a cocompact lattice.
The *Weyl chamber flow (WCF)* α_0 is the action of the subgroup

$$D_+ = \{\mathrm{diag}(e^{t_1}, \dots, e^{t_n},\ t_1 + \cdots + t_n = 0\}$$

of positive diagonal matrices on X by left translations.

We identify D_+ with the linear space

$$\{\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n,\ t_1 + \cdots + t_n = 0\}$$

- *Lyapunov exponents* are linear functionals $t_i - t_j$ on D_+.
- *Lyapunov hyperplanes* are the kernels of Lyapunov exponents, i.e. the hyperplanes. $t_i = t_j$
- *Regular elements* of D_+ are elements outside the Lyapunov hyperplanes.
- *Weyl chambers* are connected components of regular elements. Elements within each Weyl chamber have the same stable and unstable distributions and foliations.
- (Coarse)*Lyapunov distributions (foliations)* are minimal nonempty intersections of stable distributions (foliations) for different Weyl chambers.

Lyapunov foliations for WCF

- v_{ij} the elementary $n \times n$ matrix with only one nonzero entry equal to one, namely, that in the row i and the column j and $e_{ij}(t) = \exp tv_{ij}$, the corresponding unipotent one-parameter subgroup in $SL(n, \mathbb{R})$.

- $\mathcal{U}_{ij}$ the homogeneous unipotent foliation of X into left cosets of the subgroup e_{ij} so that for $x \in X = SL(n, \mathbb{R})/\Gamma$, the leaf of $\mathcal{U}_{ij}$ through x is $\mathcal{U}_{ij}^x = \{e_{ij}(t)x,\ t \in \mathbb{R}\}$.
  These foliations are *Lyapunov foliations for the WCF.*

- $\mathcal{U}_{ij}$ commutes with the orbit foliation $\mathcal{N}_0$ of WCF;
  the sum of their tangent distributions integrates to an n-dimensional foliation $\mathcal{U}_{ij}^c$.

Generic restrictions and rigidity

- Two-dimensional plane $P \subset D_+$ is *in general position* if it intersects distinct Lyapunov hyperplanes along distinct lines.
- Let $G \subset D_+$ be a closed subgroup which contains a lattice L in a plane in general position.
  The restriction $\alpha_{0,G}$ of WCF to such G is called *generic*.
- One can naturally think of G as the image of an injective homomorphism $i_0 : \mathbb{Z}^k \times \mathbb{R}^l \to D_+$. where $k + l \geq 2$.

We prove LOCAL RIGIDITY FOR GENERIC RESTRICTIONS:

Any perturbation of a generic restriction for a WCF sufficiently small in C^2 topology is differentiably conjugate to a standard one which appears from a homomorphism $i : \mathbb{Z}^k \times \mathbb{R}^l \to D_+$ close to i_0.

Results for G actions follow easily from those for L actions.

Strategy of proof

Preliminary steps

1. Reduction to Hölder perturbations in the orbit direction for WCF;
2. Reduction to the problem of trivialization of cocycles over perturbations.

Main step

Rigidity of Hölder cocycles over the perturbed action.

Final step

Smoothness of transfer function for smooth cocycles and smoothness of the conjugacy.

Reduction to Hölder perturbations in the neutral direction

- $\tilde{\alpha}_G$ a C^2-small perturbation of the generic restriction $\alpha_{0,G}$.
- Regular elements of $\alpha_{0,G}$ are normally hyperbolic to the orbit foliation $\mathcal{N}_0$ of the WCF.
- By the Hirsch-Pugh-Shub stability theory, the corresponding element of the perturbed action is normally hyperbolic to a Hölder foliation $\mathcal{N} = h'\mathcal{N}_0$ with smooth leaves;
- h' is a Hölder homeomorphism close to id it can be chosen smooth and C^2 close to id along the leaves of $\mathcal{N}_0$.
- By commutativity $\mathcal{N}$ is $\tilde{\alpha}_G$ invariant.
- $\alpha_G := h'^{-1} \circ \alpha_{0,G} \circ h'$ is a perturbation of $\alpha_{0,G}$ which preserves the leaves of $\mathcal{N}_0$ and is C^2-small along the leaves of $\mathcal{N}_0$.
- *A priori*, α_G is only Hölder transversally.

Reduction to cocycle rigidity for perturbations

- A perturbation α_G of $\alpha_{0,G}$ along $\mathcal{N}_0$ is given by a map $\beta : \mathbb{G} \times X \to \mathbb{D}_+ :\ \alpha_G(a, x) := diag(\exp \beta(a, x)) \cdot x.$
- β is a *cocycle* over α_G, i.e. for $a, b \in \mathbb{G}$ and $x \in X$ $\beta(a + b, x) = \beta(a, \alpha_G(b, x)) + \beta(b, x).$
- If β is cohomologous to a constant cocycle $i : \mathbb{G} \to \mathbb{D}_+$ i.e. $\beta(a, x) = H(\alpha_G(a, x)) + i(a) - H(x),$ then the map $H : X \to \mathbb{D}_+$ induces a conjugacy $h(x) := H(x)^{-1} \cdot x$ between α_G and $\alpha_{0,G} \circ i.$
- Thus Hölder conjugacy between $\tilde{\alpha}_G$ and $\alpha_{0,G}$ follows from HÖLDER COCYCLE RIGIDITY:

($\mathfrak{C}$) *Every Hölder cocycle over a sufficiently small perturbation of* $\alpha_{0,G}$ *in the neutral direction, which is conjugate to a* C^2 *small perturbation is Hölder cohomologuos to an injective homomorphism* i.

Cocycle rigidity for generic restrictions

The proof of ($\mathfrak{C}$) is both motivated and based upon our proof of rigidity of Hölder cocycles for generic restrictions of WMF in

D. Damjanović, A. Katok, Periodic cycle functionals and Cocycle rigidity for certain partially hyperbolic $\mathbb{R}^k$ actions, Discr. Cont. Dyn.Syst., 13, (2005), 985–1005.

Unlike most earlier proofs of cocycle rigidity for hyperbolic or partially hyperbolic actions of higher rank abelian groups our proof does not use harmonic analysis. It relies upon

- the *geometry of Lyapunov foliations* and.
- the *classical description of generators and relations in* $SL(n, \mathbb{R})$.

- The action $\alpha_{0,G}$ is locally Hölder transitive: any two ϵ-close points on X can be connected by a broken path whose pieces lie in the leaves of the Lyapunov foliations (Lyapunov paths), and the length of the path is comparable to a positive power of ϵ.
- *Coarse Lyapunov foliations* for the action $\alpha_{0,G}$ are the same as for the whole WCF: one-dimensional unipotent foliations $\mathcal{U}_{ij}$.
- For every partially hyperbolic diffeomorphism with the property that stable and unstable foliations are locally transitive one can describe a complete system of obstructions to trivialization of Hölder cocycles in terms of the *periodic cycle functionals* (PCF); A.K.–A.Kononenko. This modifies to actions of abelian groups using Lyapunov periodic paths.

Let α be an $\mathbb{R}^k$ action with the coarse Lyapunov foliations $\mathcal{F}_1, .., \mathcal{F}_r$. Let $\beta : \mathbb{R}^k \times M \to \mathbb{R}$ be a Hölder cocycle over α. For $a \in \mathbb{R}^k$ denote the function $\beta(a, \cdot) : M \to \mathbb{R}$ by f_a.

Let $j \in \{1, .., r\}$, $a \notin Ker\chi_j$, $y \in \mathcal{F}_j(x)$ and β be a Hölder cocycle cocycle over α. The *β-potential of y with respect to x* is defined by

$$P_a^j(x, y)(\beta) = * \sum^{*} f_a(a^k x) - f_a(a^k y), \tag{1}$$

where $ax := \alpha(a, x)$, $* := *(j, a) := -sgn(\chi_j(a)) \in \{+, -\}$,

$$\sum^{+} := \sum_{k=0}^{\infty}, \text{ and } \sum^{-} := \sum_{k=-1}^{-\infty}.$$

Given an $\mathcal{F}_{1,..,r}$-cycle $\mathcal{Q}$ of length N, $x_1, \ldots, x_N, x_{N+1} = x_1$ where for each $i = 1, .., N$, $j(i) \in \{1, .., r\}$ is such that $x_{i+1} \in \mathcal{F}_{j(i)}(x_i)$. define the *periodic cycle functional* on the space of Hölder cocycles over α by

$$F_a(\mathcal{Q})(\beta) = \sum_{i=1}^{N} P_a^{j(i)}(x_i, x_{i+1})(\beta)$$

Vanishing al the PCF on all cycles is equivalent to Hölder cohomology to a constant cocycle.

THE KEY STEP:

The obstructions vanish on contractible Lyapunov cycles due to

- The invariance of the periodic cycle functional and,
- The algebraic structure of $SL(n,\mathbb{R})$:

  Presentation of the group found by Steinberg and

  Matsumoto's presentation of the kernel of its universal central extension which allowed Milnor to descrbie of continuous continuous K_2 over the real and complex field.

The group $SL(n,\mathbb{R})$ is generated by unipotent elements $e_{ij}(t), t \in \mathbb{R}, 1 \leq i \neq j \leq n$ subject to standard commutator relations:

$$e_{ij}(t)e_{ij}(s) = e_{ij}(t+s) \tag{2}$$

$$[e_{ij}(t), e_{kl}(s)] = \begin{cases} 1, & j \neq k, i \neq l \\ e_{il}(st), & j = k, i \neq l \\ e_{kj}(-st), & k \neq j, i = l \end{cases} \tag{3}$$

where $[\cdot,\cdot]$ denotes the commutator; and the following extra relations:

$$h_{12}(t)h_{12}(s) = h_{12}(ts) \tag{4}$$

where $h_{12}(t) := e_{12}(t)e_{21}(-t^{-1})e_{12}(t)e_{12}(-1)e_{21}(1)e_{12}(-1)$ for $t \in \mathbb{R}^*$.

- Each one of the relations in (**??**) and (**??**) induces a $\mathcal{U}$-path at a point $x \in X$. Such paths we call *commutator* paths.

  Any commutator path lies inside a single stable leaf of a certain element of the WCF. For this reason, such paths we also call *stable* paths for the WCF and hence for a generic restriction.

- By invariance the PCF vanishes on all stable paths.

- Vanishing of the PCF on the paths defined by (**??**) follows from Milnor's description of continuous *Steinberg symbols*.

- Thus PCF vanishes on all contractible paths and defines a homomorphism $\Gamma \to \mathbb{D}_+$

  Due to Margulis' normal subgroup theorem every homomorphism from Γ into $\mathbb{D}_+$ vanishes;
  this implies vanishing of obstructions on all Lyapunov paths.

Hölder cocycle rigidity for perturbations

- *Persistence of local transitivity* of strong stable and unstable foliations under C^2-small perturbations for partially hyperbolic diffeomorphisms is a classical result of Brin and Pesin; this extends straightforwardly to (coarse) Lyapunov foliations for partially hyperbolic actions.

- *Description of obstructions to cocycle trivialization* in terms of the periodic cycle functional on closed Lyapunov paths for the perturbed action $\tilde{\alpha}_G$, and consequently for its conjugate α_G .

- *Continuous canonical projections* of Lyapunv paths along the leaves of $\mathcal{N}_0$ are defined for $x \in X$ and $y \in \mathcal{N}_0^x$:
  $\mathcal{P}_{x,y} : \mathcal{C}\mathcal{U}_x \to \bar{\mathcal{U}}_y$ and $\bar{\mathcal{P}}_{x,y} : \mathcal{C}\bar{\mathcal{U}}_x \to \mathcal{U}_y$.
  Here $\mathcal{C}\bar{\mathcal{U}}_x$ is the collection of all closed Lyapunov paths for α_G starting at $x \in X$; $\mathcal{C}\bar{\mathcal{U}} = \cup_{x \in X} \mathcal{C}\bar{\mathcal{U}}_x$;
  $\bar{\mathcal{U}}_x$ and $\bar{\mathcal{U}}$ are corresponding collections of Lyapunov paths which are not necessarily closed;
  $\mathcal{C}\mathcal{U}_x$, $\mathcal{C}\mathcal{U}$, $\mathcal{U}_x$, $\mathcal{U}$ are the corresponding collections of Lyapunov paths for the unperturbed action $\alpha_{0,G}$.
- Using the structure of closed contractible Lyapunov paths in $\mathcal{C}\mathcal{U}$ and continuity of $\bar{\mathcal{P}}_{x,y}$ and of 1-parameter unipotent foliations we show that $\bar{\mathcal{P}}_{x,y}(\mathcal{C}\bar{\mathcal{U}}_x) \subset \mathcal{C}\mathcal{U}_y$.
- The reverse projection $\mathcal{P}_{x,y}$ is such that for $\mathfrak{p} \in \mathcal{C}\bar{\mathcal{U}}_x$, $\mathcal{P}_{x,y}(\mathfrak{p})$ has endpoints on the same leaf of $\mathcal{N}_0$. This defines a map $\mathcal{D} : \mathcal{C}\bar{\mathcal{U}} \to D_+$.

- $\mathcal{D}$ is invariant along the leaves of Lyapunov foliations for the perturbation.
- By the local transitivity of those foliations, $\mathcal{D}(\mathcal{C}\bar{\mathcal{U}}_x)$ is a *closed subgroup* of D_+ which does not depend on x.
- This subgroup must be discrete due to the smallness of the perturbation and due to the fact that $\bar{\mathcal{P}}$ takes closed paths to closed paths. Thus PCF for $\tilde{\alpha}_G$ vanishes on all contractible paths.
- *Obstructions for cocycle trivialization* vanish for the perturbed action by similar arguments as before for the linear action $\alpha_{0,G}$.

Smoothness of the conjugacy

A Hölder map conjugating the action $\alpha_{0,G}$ *to its* C^2*-small* C^∞ *perturbation* $\tilde{\alpha}_G$*, is smooth.*

The transversal smoothness of the conjugacy follows from the non-stationary normal forms method and the rigidity of centralizers for extensions (M. Guysinsky–A.K).

Global smoothness is a consequence of the general fact that a function smooth along several smooth foliations whose tangent distributions with their Lie brackets generate the tangent space, is necessarily smooth.