# THE 1977 WARSAW CONFERENCE, COCYCLES OVER $\mathbb{Z}^k$ ACTIONS, AND RIGIDITY PROGRAM

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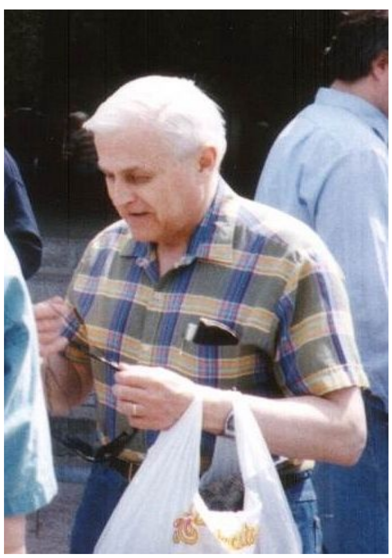
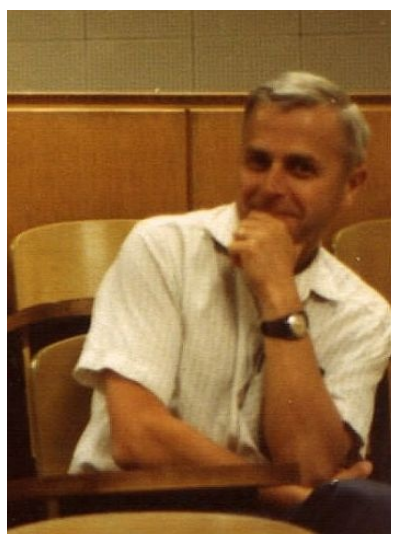
## 1. Warsaw conference and the role of Wieslaw Szlenk

Wieslaw Szlenk was the prime mover and the principal organizer of the 1977 Warsaw conference in dynamical systems. It was a pivotal event in the development of modern dynamics and the formation of the world-wide community which still after 28 years keeps many of its principal features. It was the place, probably almost the only one possible under the Cold War conditions prevailing at the time, for "East" to meet "West", for the leading researchers from the US and the Soviet Union in their prime (still mostly quite young) to meet each other in force and often for the first time, and to meet also with their up-and-coming European colleagues. I will share some recollections and show some images related to that conference.





This is Wieslaw Szlenk on a Warsaw street and in the conference lecture room

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And this is the famous (or infamous for some) Palace of Culture,
the conference venue, where mathematicians are not present anymore

## 2. Special representation of $\mathbb{R}^k$ actions

It so happened that at the conference I spoke on the subject of the representation of $\mathbb{R}^k$ actions via cocycles over $\mathbb{Z}^k$ actions, a multi-dimensional generalization of the classical Ambrose–Kakutani theorem about representation of flows as special flows. At the time my motivation came from my previous work, the development of what is known now as Kakutani equivalence theory which I called monotone equivalence. That was my first and for many years to come the only serious encounter with actions of higher rank abelian groups, then still mostly thought of as a rather benign extension of the classical theory, rather than fascinating world of its own. My approach was to start from a section which is discrete and cocompact, i.e whose intersections with an $\mathbb{R}^k$ orbit form a net and to extend it to one which has a lattice structure with bounded distortion with the respect to the time parameters. At the time it was not known whether such a structure could be put on any discrete and cocompact section, a question asked by Furstenberg, of which I was unaware, and which was solved in the negative twenty years later by Burago, Ivanov and Kleiner and independently by McMullen.

The question of existence of a lattice structure for sections of particular actions, such as Weyl chamber flows, turned out to be related to the rigidity properties of actions of higher rank abelian groups, which has been the primary topic of my research interests during the last decade. Cocycle rigidity indicates that one should not expect existence of such a structure because then the distortion would be straightened out and the action would have to be a suspension which is incompatible with mixing. With some extra conditions on a section this statement can be made precise.

I will discuss connections between the higher–dimensional special representations and rigidity and in particular describe a specific problem if that kind which is related to a possibilty of a certain version of the multi-dimensional continued fractions algorithm.

## 3. Nets, lattices and sections for $\mathbb{R}^k$ actions

A subset $N \subset \mathbb{R}^k$ is a *separated net*, if for some positive numbers $r < R$ every ball of radius $r$ contains at most one point form $N$ and every ball of radius $R$ contains at least one point from $N$. Let $\alpha$ be a Borel (or measure-preserving) action of $\mathbb{R}^k$ on a Borel (or Lebesgue) space $X$, A Borel set $S \subset X$ is called a *uniform section* if the intersection of $S$ with any $\alpha$ orbit is a separated net. See [4] for the corresponding definition in the measure-preserving case.

The question of whether any separated net is bi-Lipschits equivalent to the lattice $\mathbb{Z}^k$ is a particular case of a general question posed by Gromov. It was answered in the negative in 1998 independently by Burago and Kleiner [2] and by McMullen [5]. The corresponding question for the $\mathbb{R}^k$ actions is whether a uniform section possesses a Borel (or measurable) $\mathbb{Z}^k$ structure bi-Lipshtitz along the orbits. Apparently this question was raised by Furstenberg in the sixties long before Gromov. Notice that in this case even existence of a section with such a $\mathbb{Z}^k$ structure is not immediately apparent if $k \geq 2$, and was proved in [4].

Burago-Kleiner construction can be adapted to provide a counter-example to the Furstenberg question; however, in such an adaptation the action is specially constructed simultaneously with the section. In other words, such examples are not "natural".

We provide a class of natural examples of actions and sections without natural lattice structures. We cannot show however that no measurable lattice structure exist; thus so far our examples do not provide natural counter-examples to the Furstenberg question. Unlike the Burago-Kleiner and McMullen methods which are of analytic nature our method is based on geometry and ergodic theory of simple Lie groups. We restrict ourselves to the most basic case where the Lie group in question is $SL(n, \mathbb{R})$.

## 4. The main result

4.1. **Formulations.** Let $n \geq 3$ and let $\Gamma$ be a co-compact lattice in $SL(n, \mathbb{R})$. Recall that *Weyl chamber flow* is the action of the subgroup $D_+$ of positive diagonal matrices of the homogeneous space $SL(n, \mathbb{R})/\Gamma$ by left translations. For $1 \leq i \neq j \leq n$ we denote by $\mathcal{U}_{ij}$ the one-dimensional homogeneous foliation into the left cosets of the one-parameter unipotent group $\exp tN_{ij}$, $t \in R$, where $N_{ij}$ is the matrix with one at the intersection of the $i$th row and the $j$th column and zeroes elsewhere.

A $\mathbb{Z}^k$ structure on a section $S$ of an $\mathbb{R}^k$ action $\alpha$ is a $\mathbb{Z}^k$ valued cocycle on the equivalence relation produced by intersecting $S$ with the orbits of $\alpha$. In other words, let $A \subset S \times S$ be the set of pairs which lie on the same $\alpha$ orbit. Then one has a map

$$\beta : A \to \mathbb{Z}^k$$

such that

$$\beta(x, y) + \beta(y, z) = \beta(x, z) \quad \text{and} \quad \beta(y, x) = -\beta(x, y). \tag{1}$$

If $x$ has trivial stationary subgroup with respect to $\alpha$ then of course any point $y$ on the $\alpha$ orbit of $x$ can be represented as $\alpha(x, t)$ with uniquely defined $t \in \mathbb{R}^k$. For the actions which we consider such as the Weyl chamber flow all but countably many orbits have trivial stationary subgroups but the others are compact and their union is dense. Since we will prove non-existence of $\mathbb{Z}^k$ structures we will ignore those compact orbits.

**Theorem 1.** *Let $S$ be a piecewise smooth uniform section of the Weyl chamber flow with the extra property that for some $r > 0$ its intersection with any leaf of every foliation $\mathcal{U}_{ij}$ does not contain two points at the distance less than $r$ in the inner metric of the leaf. Then $S$ does not possess a piece-wise constant (or, which is the same piecewise continuous) $\mathbb{Z}^k$ structure bi-Lipshtitz along the orbits of the Weyl chamber flow.*

The main ingredient in the proof is a modification of the cocycle rigidity results for Hölder cocycles over Weyl chamber flows and related actions from [3] to piecewise Lipschitz cocycles.

## References

[1] D. Bernstein, A. Katok and G.D. Mostow, Nets and quasi-isometries, preprint, 1985, unpublished.

[2] D. Burago and B. Kleiner, Separated nets in Euclidean space and Jacobians of bi-Lipschitz maps. *Geom. Funct. Anal.* **8** (1998), no. 2, 273–282.

[3] D. Damjanovic and A. Katok, Periodic cycle functionals and cocycle rigidity for certain partially hyperbolic $\mathbb{R}^k$ actions *Discrete and Continuous Dynamical Systems*, **13** (2005), 985-1005

[4] A. Katok, The special representation theorem for multi-dimensional group actions, Dynamical Systems, I, Warsaw,*Asterisque* **49** (1977), 117-140.

[5] C. McMullen, Lipschitz maps and nets in Euclidean space. *Geom. Funct. Anal.* **8** (1998), no. 2, 304–314.

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