WEAKLY MIXING BILLIARDS

by

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1. Introduction.

In this paper we make a modest new contribution to the study of dynamical properties of polygonal billiards using categorial approach.

In very general terms, the approach is based on Baire category theorem and on an approximation principle which says that if a Baire space B has a dense set of elements satisfying an approximate version of a certain property then it contains a dense G_δ set of elements which possess that property exactly. Without trying to discuss here what properties can be studied that way, we refer to [K1] where categorial approach is developed systematically for various spaces of dynamical systems. In a number of cases categorial approach or its modification provide the only known way to establish the existence of dynamical systems with a particular property. Existence of ergodic billiards [KMS], [K2], discussion below, is a good example of such a situation.

Let P be a connected polygon in Euclidean plane $\mathbb{R}^2$. The billiard flow B_P^t is defined on the space Y_P of all unit tangent vectors to $\mathbb{R}^2$ with footpoints in P. It can be described as follows. A vector $v \in Y_P$ with the footpoint $p \in P$ moves with the unit speed along the straight line $p + vt$, $t \in \mathbb{R}$ until it reaches the boundary of P, then it instantly changes its direction according to the rule "the angle of incidence is equal to the angle of reflection" and continues until the next collision with the boundary and so on. If a vector hits a vertex of P, the flow is not defined after the collision. The billiard flow thus defined preserves the Liouville measure on Y_P which is the product of Lebesgue measure on P and the angular measure on the circle of directions. The set of vectors which eventually hit a vertex of P has Liouville measure zero so that from the point of view of ergodic theory the billiard flow is well defined.

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The phase space Y_P of the billiard flow is three-dimensional and in general very little is known about ergodic properties of that flow. About the only general statement of that kind is that the entropy of B_P^t is equal to zero. This is true not only for the Liouville measure but for any Borel invariant measure as well [K3]. However for certain classes of polygones more information is known. A polygon P is called *rational* if all of its angles are commensurate with π. For any rational polygon P the space Y_P splits into a one-parameter family of two-dimensional subsets $Y_{P,\theta}$, $0 \le \theta < \frac{\pi}{N(P)}$ invariant with respect to the billiard flow [ZK], [G]. Here N(P) is the least common multiple of the denominators of the numbers $\frac{\alpha}{\pi}$ where α runs over the set of angles of P. By appropriate identification, the set $Y_{P,\theta}$ is made into a compact surface. Let us denote the restriction of B_P^t to $Y_{P,\theta}$ by $B_{P,\theta}^t$ and call it the *directional billiard flow*. The number of ergodic invariant measures for such a flow, which are not supported by periodic orbits, is bounded [S]. The flow $B_{P,\theta}^t$ is not mixing [K4]. A recent fundamental result [KMS] says that for almost every θ the flow $B_{P,\theta}^t$ is uniquely ergodic. When the number N(P) becomes large, the surfaces $Y_{P,\theta}$ become more and more uniformly distributed in Y_P. This sets the stage for the application of the categorial approach [K2], [KMS] which allows in particular to establish the existence of billiard flows ergodic in the whole space Y_P. This argument mimics an earlier similar argument [ZK] related to topological transitivity. It is still not known whether for a generic rational polygon P for most θ the flows $B_{P,\theta}^t$ are weakly mixing. "Most" may mean either a set of full measure or a dense G_δ. In this paper we solve this question in the sense of category for certain classes of rational polygons. Namely, we consider polygons for which the number N(P) is equal to 2, 3, 4 or 6. Each of these classes contains a dense subset of so-called almost integrable polygons (see Definition 3 below) which do have non-constant eigenfunctions [G]. Within our classes the almost integrable polygons are characterized by rational values of some natural parameters. When denominators of those parameters go to infinity, the non-constant eigenfunctions become more and more oscillating and eventually disappear for polygons with irrational but very well approximable values of the parameters.

An interesting open problem is the existence and genericity of billiards which are weakly mixing in whole phase space Y_P. Let us fix the topology of the billiard table P, i.e., the number of connected components of the boundary of P and the number of vertices on each boundary component. Let n be the total number of vertices. Let $\tilde{P}$ be the space of all such billiard tables with topology given by parametization by the coordinates of vertices. $\tilde{P}$ is a non-compact manifold of dimension 2n.

Theorem [KMS], [K2]. The set of all polygons $P \in \tilde{P}$ such that the billiard flow B_P^t is ergodic is a dense G_δ subset of $\tilde{P}$.

It is not difficult to see that the set $\tilde{P}_{mix}$ of all $P \in \tilde{P}$ for which B_P^t is weakly mixing is a G_δ.

Conjecture. The set $\tilde{P}_{mix}$ is a dense G_δ subset of $\tilde{P}$.

2. Preliminaries. Statement of Results.

For any polygon P we denote by U_P^t the one-parameter group of unitary operators on $L_2(Y,\mu)$ corresponding to the billiard flow B_P^t. Here μ is the (unnormalized) Liouville measure on Y_P. We assume $\mu(Y_P) = |P|$, which is the area of P.

The group G generated by Euclidean motions and dilations of the plane acts naturally on $\tilde{P}$ and the quotient $\tilde{P}/G$ can be identified with the submanifold P of $\tilde{P}$ consisting of polygons P with a distinguished vertex at the origin of $\mathbb{R}^2$, the first side on the positive x-axis and $|P| = 1$. Clearly, $\dim P = 2n - 4$ and, because the action of G is compatible with the flows B_P^t, it suffices to study those flows for $P \in P$.

We identify the set of directions θ on the plane with the circle $S^1 = \{0 \le \theta < 2\pi\}$ where $\theta = 0$ corresponds to the direction of the positive x-axis.

Definition 1. A polygon is called *integrable* if it tiles the plane under reflections.

It is well known that the only integrable polygons are rectangles, the equilateral triangles, the $\pi/2$, $\pi/4$, $\pi/4$-triangles, and the $\pi/2$, $\pi/3$, $\pi/6$-triangles.

We fix an integrable polygon Δ and denote by Γ the lattice obtained by tiling the plane by reflections of Δ. For instance, if Δ is the unit square, Γ is the square lattice.

Definition 2. A polygon $P \in \mathcal{P}$ is of *Δ-class* if the sides of P are parallel to the lines of Γ.

For instance, if Δ is a rectangle, P of Δ-class means that the sides of P are either horizontal or vertical. In what follows we denote by $\mathcal{P}$ the set of polygons of Δ-class (Δ is fixed) satisfying the previous assumptions.

Polygons $P \in \mathcal{P}$ are rational, i.e., their angles are rational multiples of π, hence, as we mentioned before, the flow B_P^t decomposes into the one-parameter family of directional billiard flows $B_{P,\theta}^t$ [Z-K], $0 \le \theta \le \pi/N(\Delta)$, where $N(\Delta) = 2, 3, 4$ or 6 depending on the type of Δ (see above). The flows $B_{P,\theta}^t$ for $0 < \theta < \pi/N(\Delta)$ live on the surface S_P, which is tiled by $2N(\Delta)$ copies of P, and preserve the Lebesgue measure μ on S_P.

Let e,f be a pair of generators of Γ. A direction θ is called *irrational* (resp. *rational*) if for a vector pe + qf in direction θ the ratio p/q is irrational (resp. rational). The definition does not depend on the choices involved.

Definition 3 [G]. A polygon $P \in \mathcal{P}$ is called *almost integrable* if it is homothetical to a polygon drawn on the lattice Γ.

The set $\mathcal{P}_I$ of almost integrable polygons is dense in $\mathcal{P}$. For an almost integrable polygon P the flow $B_{P,\theta}^t$ is ergodic if θ is irrational and periodic if θ is rational [G].

By *combinatorics* of a connected polygon P we will mean the following: The number of connected components of the boundary of P, the number of vertices and the angle at each vertex. Let n be the total number of vertices for polygons in $\mathcal{P}$.

Now we can formulate the first main result of this paper.

Theorem 1. Let Δ be an integrable polygon and let $\mathcal{P}$ be the manifold of polygons of Δ-class with fixed combinatorics. For any direction θ

denote by $P_{mix}(\theta) \subset P$ the set of polygons P such that the flow $B^t_{P,\theta}$ is weakly mixing. Then

i) Let Δ be a rectangle and $n > 4$. For any $\theta \neq 0, \pi/2$ the set $P_{mix}(\theta)$ is a dense G_δ in P.

ii) Let Δ be a triangle and $n > 3$. For any irrational direction θ the set $P_{mix}(\theta)$ is a dense G_δ.

Definition 4. A polygon M with 2n sides $a_1, b_1, \ldots, a_n, b_n$ is called *matched* if there are n parallel translations $g_1, \ldots, g_n$ such that $b_j = g_j a_j$, $j = 1, \ldots, n$.

For every direction θ we define the *linear flow* $L^t_{M,\theta}$ in direction θ on M as follows. A point in M flows in direction θ with the unit speed until it reaches the boundary of M. If this happens on a_j (resp. b_j), the point gets transferred to the side b_j (resp. a_j) by the translation g_j (resp. g_j^{-1}) and continues to move in the same direction. The Lebesgue measure on M is preserved by the flows $L^t_{M,\theta}$.

In what follows we normalize our matched polygons M by requiring that the sides a_1, b_1 be horizontal.

Definition 5. A matched polygon M is called *elementary of type* α, $0 < \alpha \leq \pi/2$, if M has only horizontal sides and sides making angle α with the x-axis.

We fix α and denote by M the set of elementary matched polygons of type α with a fixed number 4n of sides and a fixed combinatorics. The set M endowed with its natural topology is a manifold. The following theorem is a close counterpart of Theorem 1.

Theorem 2. Let $n > 1$. For any direction θ the set $M_{mix}(\theta)$ of polygons $M \in M$ such that the flow $L^t_{M,\theta}$ is weakly mixing is a dense G_δ.

The reader should keep in mind that an elementary matched polygon M with 4n sides can have less than 4n geometric vertices. A gnomon, for instance, has 8 sides and 6 geometric vertices. In other words, some of the angles of M may be equal to π.

Theorems 1 and 2 are derived via categorial approach from a result which describes the discrete spectrum of linear flows in almost integrable polygons. We need more definitions to state the corresponding theorem.

Definition 6. We say that an elementary matched polygon M of type α *is modelled on the parallelogramm* A if M is tiled by translated copies of A and A is a maximal parallelogramm to tile M. In what follows we simply say that *M is modelled on* A.

Definition 7. Let A be a parallelogramm spanned by e and f. A direction θ is called *irrational* (resp. *rational*) *with respect to* A if for a vector ae + bf in direction θ the number a/b is irrational (resp. rational).

Let M be modelled on A and let A_i, $i \in I$, be the copies of A tiling M. For any $i \in I$ we identify functions on A_i and A.

Definition 8. Let notation be as above. A function f on M is called A-*periodic* if the restrictions of f on A_i, $i \in I$, are all equal.

We denote by $L_2^{(d)}(M)$ the Hilbert space of square integrable (with respect to the Lebesgue measure) A-periodic functions on M. By a natural isomorphism, $L_2^{(d)}(M) = L_2(A)$.

Theorem 3 (cf. [G], Theorem 3). Fix a parallelogram A and let M be a polygon modelled on A. Then

i) The flow $L^t_{M,\theta}$ is uniquely ergodic if θ is irrational and periodic otherwise.

ii) For any irrational direction θ the discrete spectrum component of $L_2(M)$ for the flow $L^t_{M,\theta}$ is the space $L_2^d(M)$ of A-periodic functions. The identification $L_2^d(M) = L_2(A)$ induces a natural isomorphism of $L^t_{M,\theta}$ restricted to $L_2^d(M)$ with $L^t_{A,\theta}$.

Consider the space $\mathcal{Q} = \mathcal{P} \times S^1$ of pairs (P,θ) where $\mathcal{P}$ is the space of polygons of Δ-class. We want to show that for a typical pair (P,θ) the flow $B^t_{P,\theta}$ is weakly mixing.

Theorem 4. Let $\mathcal{P}$ be the space of polygons of Δ-class (Δ is fixed) and let $\mathcal{Q}_{mix}$ be the set of pairs $Q = (P,\theta)$ such that $B^t_{P,\theta}$ is weakly mixing. Then $\mathcal{Q}_{mix}$ is a dense G_δ in $\mathcal{Q}$.

3. Proofs.

Proof of Theorem 3. Let M be any matched polygon with the pairs a_i, b_i, $i = 1, \ldots, n$, of parallel sides. Identifying a_i with b_i for all i we obtain a closed surface S_M and the flows $L^t_{M,\theta}$ live on S_M.

The surface S_A corresponding to a parallelogramm A is a torus and the flow $L^t_{A,\theta}$ is the linear flow in direction θ on the torus S_A.

The tiling of M by copies of A defines the projection $p:S_M \to S_A$ which commutes with the flows $L^t_{M,\theta}$ and $L^t_{A,\theta}$ for all θ. Now we are in the setting of Theorem 3 of [G] and we refer the reader to the proof of that theorem.

Proof of Theorem 2. We consider polygons $M \in \mathcal{M}$ of area one such that (0,0) is a vertex of M and obviously it suffices to prove the assertion for the manifold (denoted again by $\mathcal{M}$) of polygons satisfying these conditions.

Fix a direction θ and choose a parallelogramm A_θ, $|A_\theta| = 1$, with angle α such that θ is irrational with respect to A_θ. Denote by (x,y) the linear coordinates defined by A_θ so that $A_\theta = \{(x,y):0 \le X \le 1,\ 0 \le y \le 1\}$. Let e_0 and f_0 be the vectors spanning A_θ. Denote by $\mathcal{A}$ the set of parallelogramms A spanned by $e = re_0$ and $f = sf_0$ where r and s are rational and let $\mathcal{M}_I \subset \mathcal{M}$ be the subset of polygons modelled on A, $A \in \mathcal{A}$. Let a_i, b_i and c_i, d_i, $i = 1, \ldots, n$, be respectively the pairs of horizontal sides and the sides forming angle α with horizontal direction. Then $M \in \mathcal{M}_I$ if and only if the numbers $|a_i|/|e_0|$, $|c_i|/|f_0|$ are rational for $i = 1, \ldots, n$. Here the absolute value sign denotes the length of a vector. From now until the end of the proof we delete θ from notation.

We denote by 1_M the indicator function M. We have the natural embedding $L_2(M) \to L_2(\mathbb{R}^2)$ and the projection $L_2(\mathbb{R}^2)$ on $L_2(M)$ given by $f \to 1_M f$. Using this we extend the flows L^t_M and the unitary groups U^t_M to $\mathbb{R}^2$ and $L_2(\mathbb{R}^2)$ respectively by identity on $\mathbb{R}^2 \setminus M$. We use the same symbols for the extended L^t_M and U^t_M and denote by $\langle f,g \rangle$ the scalar product in $L_2(\mathbb{R}^2)$.

The flow L^t_M is weakly mixing if for any $f \in L_2(\mathbb{R}^2)$ the function $t \to \langle U^t_M f, 1_M f \rangle$ strongly converges in the sense of Cesaro (see [H] or [W]) to $|\langle f, 1_M \rangle|^2$ as $|t| \to \infty$. We will need the following.

Lemma 1. For any $f,g \in L_2(\mathbb{R}^2)$, any t and any $\varepsilon > 0$ the set of polygons $M \in \mathcal{M}$ such that

$$|\langle U_M^t f, 1_M g\rangle - \langle f, 1_M\rangle\langle 1_M, g\rangle| < \varepsilon \qquad (1)$$

is open in $\mathcal{M}$.

Proof. For any t, f and g the functions $M \to \langle f, 1_M\rangle$ and $M \to \langle U_M^t f, 1_M g\rangle$ are continuous on $\mathcal{M}$. □

We choose a dense in $L_2(\mathbb{R}^2)$ sequence f_i, i = 1, 2 ... and for any t and N ⩾ 1 denote by $\mathcal{M}_{t,N} \subset \mathcal{M}$ the set of polygons M such that for i = 1, ..., N

$$|\langle U_M^t f_i, 1_M f_i\rangle - |\langle f_i, 1_M\rangle|^2| < 1/N. \qquad (2)$$

In view of Lemma 1, $\mathcal{M}_{t,N}$ is open in $\mathcal{M}$ for any t and N and we set $\mathcal{M}_N = \bigcup_t \mathcal{M}_{t,N}$. Thus, $\mathcal{M}_N$ is open and $\bigcap_{N=1}^{\infty} \mathcal{M}_N$ is a G_δ. We claim that $\mathcal{M}_{mix} = \bigcap_N \mathcal{M}_N$.

If $M \in \mathcal{M}_{mix}$ then (cf. [H] or [W]) for any $f, g \in L_2(\mathbb{R}^2)$ there is a set $T_{f,g}$ of density one in $\mathbb{R}$ such that $\langle U_M^t f, 1_M g\rangle$ converges to $\langle f, 1_M\rangle\langle 1_M, g\rangle$ when $|t| \to \infty$ in $T_{f,g}$. Intersection of a finite number of sets of density one has density one, hence nonempty, therefore $\mathcal{M}_{mix} \subset \bigcap_N \mathcal{M}_N$.

Assume that the opposite inclusion fails, i.e., that there exists $M \in (\bigcap_N \mathcal{M}_N)\setminus\mathcal{M}_{mix}$. Then there is an eigenfunction $f \in L_2(M)$ of U_M^t such that $\langle f, 1_M\rangle = 0$ and $\|f\| = 1$. Let $U_M^t f = \exp(\sqrt{-1}\,at)f$. Fix $\varepsilon > 0$ and let f_i be such that $\|f - f_i\| < \varepsilon$. For any $t \in \mathbb{R}$

$$U_M^t f_i = U_M^t(f_i - f) + U_M^t f = U_M^t(f_i - f) + \exp(\sqrt{-1}\,at)f.$$

Therefore

$$\langle U_M^t f_i, 1_M f_i\rangle = \langle U_M^t(f_i - f) + \exp(\sqrt{-1}\,at)f,\ 1_M(f_i - f) + f\rangle$$

$$= \langle U_M^t(f_i - f), 1_M(f_i - f)\rangle + 2\exp(\sqrt{-1}\,at)\mathrm{Re}\,\langle (f_i - f), f\rangle + \exp(\sqrt{-1}\,at)$$

which implies the estimate

$$|\langle U_M^t f_i, 1_M f_i\rangle - \exp(\sqrt{-1}\ at)| < 2\varepsilon + \varepsilon^2. \tag{3}$$

Since $|\langle f_i, 1_M\rangle| = |\langle f_i - f, 1_M\rangle| < \varepsilon$, we have for any t

$$|\langle U_M^t f_i, 1_M f_i\rangle - |\langle f_i, 1_M\rangle|^2| > |\langle U_M^t f_i, 1_M f_i\rangle| - \tag{4}$$

$$|\langle f_i, 1_M\rangle|^2 > 1 - 2\varepsilon - 2\varepsilon^2.$$

Taking ε small enough in (4) we find an index i such that

$$|\langle U_M^t f_i, 1_M f_i\rangle - |\langle f_i, 1_M\rangle|^2| > \tfrac{1}{2} \tag{5}$$

for all t. Hence, for $N = i + 1$, $M \notin \mathcal{M}_N$ in contradiction to the assumption.

We have shown that $\mathcal{M}_{mix} = \bigcap_n \mathcal{M}_n$ is a G_δ.

It remains to show that $\mathcal{M}_{mix}$ is dense. For $M \in \mathcal{M}_I$ denote by p(M) and q(M) the least common denominators of $|a_i|/|e_0|$, $i = 1, \ldots, n$ and $|c_i|/|f_0|$, $i = 1, \ldots, n$ respectively. If $p(M) = p$ and $q(M) = q$, M is tiled by copies of the parallelogramm $A_{p,q}$ spanned by $e = e_0/p$ and $f = f_0/q$. Denote the parallelogramm $\{xe_0+yf_0 : |x|,|y| \le N\}$ by B_N.

For $M \in \mathcal{M}_I$ denote by P_M^d (resp P_M^c) the projection on the nontrivial discrete spectrum, i.e., on the discrete spectrum inside the space $L_2(M)$ (resp. continuous spectrum) of U_M^t. Let $p(M) = p$, $q(M) = q$ and let $M \subset B_N$. By Theorem 3 $(P_M^d f)(x,y) =$

$$(pq)^{-1}\ 1_M(x,y) \sum_{i=-pN}^{pN} \sum_{j=-qN}^{qN} (1_M f)(x+i/p, y+j/q). \tag{6}$$

Denote (x,y) by z and (i/p,j/q) by e_{ij}. We rewrite (6) as

$$(P_M^d f)(z) = (pq)^{-1} \sum_{i,j} (1_M f)(z+e_{ij}). \tag{7}$$

Denote $P_M^d f$ by g. Since g is $A_{p,q}$-periodic, for any $z \in M$ there exists $u \in A_{p,q}$ such that $g(z) = g(u)$. Thus, for any $z, z' \in M$ there are $u, u' \in A_{p,q}$ so that

$$g(z) - g(z') = g(u) - g(u') = (pq)^{-1} \sum_{i,j} f(u+e_{ij}) - f(u'+e_{ij}) \tag{8}$$

where the summation is over such pairs (i,j) that $e_{ij} \in M$.

Let f be a continuous function supported on B_N. For any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $|f(z)-f(z')| < \varepsilon$ if $|z-z'| < \delta(\varepsilon)$. Fix $\varepsilon > 0$ and assume that the diameter of $A_{p,q}$ is less than $\delta(\varepsilon)$. Then $|(u+e_{ij})-(u'+e_{ij})| < \varepsilon$ and, by (8), for any $z,z' \in M$

$$|g(z)-g(z')| < \varepsilon. \tag{9}$$

Integrating (9) over M we obtain that for any $z \in M$

$$|g(z) - \int_M g(\zeta)d\mu(\zeta)| < \varepsilon. \tag{10}$$

Since g is obtained from $1_M f$ by averaging

$$\int_M g(z)d\mu(z) = \int_M f(z)d\mu(z) = \langle f,1_M\rangle. \tag{11}$$

Recalling that $g = P_M^d f$ we have, by (10) and (11)

$$\|P_M^d f - \langle f,1_M\rangle 1_M\|_u < \varepsilon \tag{12}$$

where $\|\varphi\|_u = \max|\varphi(z)|$ over $z \in B_N$. Denote by $\|\varphi\|$ the L_2-norm. If φ is supported on M, $\|\varphi\| \le \|\varphi\|_u$ and, by (12)

$$\|P_M^d f - \langle f,1_M\rangle 1_M\| < \varepsilon. \tag{13}$$

We choose a dense in $L_2(\mathbb{R}^2)$ sequence of continuous functions f_i, $i = 1, 2, \ldots$ such that supp $f_i \subset B_i$ and let $\mathcal{M}_N$, $N = 1, \ldots$ be the corresponding sequence of open sets in $\mathcal{M}$ where $\mathcal{M}_{mix} = \bigcap_N \mathcal{M}_N$. For any N we can find $\delta_N > 0$ such that $|z-z^1| < \delta_N$ implies $|f_i(z)-f_i(z')| < (2N \max_{i\le N}\|f_i\|)^{-1}$ for $i = 1, \ldots, N$.

Let $M \in \mathcal{M}_I$ be contained in B_N and assume that diam $A_{p(M),q(M)} < \delta_N$. Then for any f

$$\langle U_M^t f, 1_M f\rangle - |\langle f, 1_M\rangle|^2 =$$

$$\langle U_M^t[(P_M^d f - \langle f, 1_M\rangle 1_M) + \langle f, 1_M\rangle 1_M + P_M^c f], 1_M f\rangle - |\langle f, 1_M\rangle|^2 = \tag{14}$$

$$\langle U_M^t(P_M^d f - \langle f, 1_M\rangle 1_M), 1_M f\rangle + \langle U_M^t P_M^c, P_M^c f\rangle.$$

For $i = 1, \ldots, N$ there exists a set $T_i \subset \mathbb{R}$ of density one such that $|\langle U_M^t P_M^c f_i, P_M^c f_i\rangle| < 1/2N$ if $t \in T_i$. Hence for $t \in \bigcap_{i=1}^{N} T_i$ which is nonempty, $|\langle U_M^t P_M^c f_i, P_M^c f_i\rangle| < 1/2N$ for all $i \le N$. By (13), for any $i \le N$ and any t

$$|\langle U_M^t(P_M^d f_i - \langle f_i, 1_M\rangle 1_M), 1_M f_i\rangle| < (2N \max_{i \le N} \|f_i\|)^{-1} \|1_M f_i\| \le 1/2N. \tag{15}$$

Hence, (14) implies that for $i \le N$ and $t \in \bigcap_{i=1}^{N} T_i$

$$|\langle U_M^t f_i, 1_M f_i\rangle - |\langle f_i, 1_M\rangle|^2| < 2(2N)^{-1} = 1/N \tag{16}$$

thus, $M \in \mathcal{M}_N$. Polygons

$$\{M \in \mathcal{M}_I : M \subset B_N \text{ and } \operatorname{diam} A_{p(M),q(M)} < \delta_N\}$$

are dense in the set $X_N = \{M \in \mathcal{M} : M \subset B_N\}$, thus the closure of $\mathcal{M}_N$ contains X_N. Since any polygon belongs to some X_N, $\bigcap_N \mathcal{M}_N$ is dense in $\mathcal{M}$. □

<u>Remark 1</u>. If $n = 1$, $\mathcal{M}$ consists of parallelograms and Theorem 3 applies.

Let M be a matched polygon and let S_M be the corresponding surface (see proof of Theorem 3). The surface S_M is closed and orientable and its genus $g(S_M)$ is determined by the combinatorics of M. The conformal structure on S_M induced from M is singular at the vertices if $g(S_M) > 1$. The singularities can be resolved and S_M becomes a surface of constant negative curvature [G], but we are interested in the imposed on S_M flat conformal structure (with singularities if $g > 1$). We call such surfaces *almost flat* and denote their set by S.

Everything we said so far about matched polygons M extends to the case when M has selfoverlappings by regarding M as a union of polygons belonging to different copies of $\mathbb{R}^2$ and making natural identifications. From now on we allow M to have such selfoverlappings and denote the manifold of these polygons normalized as before by M. The mapping $M \to S$ is, by definition, onto and is locally one-to-one and thus supplies S with a structure of a manifold. If $S \in S$ we denote by $L^t_{S,\theta}$ the family of linear flows on S. The following assertion is immediate from Theorem 2.

Corollary 1. Let S be the manifold of almost flat surfaces obtained from the set M of elementary matched polygons with a fixed number $4n > 4$ of sides and a fixed combinatorics. For any θ the set $S_{mix}(\theta)$ of surfaces S such that the flow $L^t_{S,\theta}$ is weakly mixing is a dense G_δ.

Proof of Theorem 1. Let P be a rational polygon and let N be the least common multiple of the denominators of the angles $\pi m_i/n_i$ of P. Reflecting P in its sides $2N - 1$ times we obtain a matched polygon M [G]. The billiard flows $B^t_{P,\theta}$ unfold into the linear flows $L^t_{M,\theta}$ on M. Although M is not uniquely determined by P, the surface S_M does not depend on any particular way of unfolding P and $S_M = S_P$, the canonical surface defined by P [G]. Thus, we obtained a continuous mapping $s: P \to S$. Denote by D_N the dihedral group of order 2N. The image of s consists of surfaces S with an action of D_N and we have $P = s^{-1}(S) = S/D_N$.

Now we apply this to the polygons of Δ-class and notice that $N = N(\Delta)$ is equal to 2, 3, 4 and 6 if Δ is a rectangle, equilateral triangle, $\pi/4$ and $\pi/6$ triangle respectively. By fixing a way of unfolding P into M we obtain a continuous injective mapping $u: P \to M$ where M consists of elementary matched polygons of type $\alpha = \pi/2$, $\pi/6$, $\pi/2$ and $\pi/6$ when $N(\Delta) = 2$, 3, 4 and 6 respectively. We fix a direction θ and delete θ from our notation. Let M_N be the sequence of open sets introduced in the proof of Theorem 2. Since u is continuous and commutes with the flows B_P^t and L_P^t on P and M respectively, $P_N = u^{-1}(M_N)$ are open and $P_{mix} = \bigcap_N P_N$. It remains to show that P_{mix} is dense in P. We consider two cases in the theorem separately.

i) If $\theta \neq 0, \pi/2$ we can choose a rectangle Δ, $|\Delta| = 1$ such that θ is irrational with respect to Δ. Let e and f be the horizontal and the vertical vectors of Δ respectively. For any r,s > 0 denote by $\Delta_{r,s}$ the rectangle spanned by re and sf and let $P_I \subset P$ be the set of polygons which can be tiled by $\Delta_{r,s}$ under reflections where r and s are rational. Clearly, P_I is a countable dense subset of P. The rest of the proof is analogous to the second part of the proof of Theorem 2. For P $\in P_I$ we define the integers p(P) and q(P) and show that for any N the polygon P belongs to P_N if p(P) and q(P) are big enough. Thus, P_N is dense in P, therefore $P_{mix} = \bigcap_N P_N$ is a dense G_δ.

ii) We can no longer vary Δ but if θ is irrational (with rspect to Δ) we can repeat the argument of i) with obvious modifications. We spare the details.

<u>Proof of Theorem 4</u>. Let $\mathbb{Q}(\theta) = P \times \{\theta\}$. Choose a countable dense in $L_2(\mathbb{R}^2 \times S^1)$ sequence $f_i(x,y;\theta)$ such that f_i continuously depend on θ and for any fixed θ the functions $f_i(x,y;\theta)$ make a dense in $L_2(\mathbb{R}^2)$ sequence. The open sets $P_N(\theta)$ defined similar to the sets M_N (cf. (2)), continuously depend on θ and $\mathbb{Q}_{mix} \cap \mathbb{Q}(\theta) = P_{mix}(\theta) = \bigcap_N P_N(\theta)$. Set $\mathbb{Q}_N(\theta) = P_N(\theta) \times \{\theta\}$ and $\mathbb{Q}_N = \bigcup_\theta \mathbb{Q}_N(\theta)$.

Since $\mathbb{Q}_N(\theta)$ is open in $\mathbb{Q}(\theta)$ for any θ and depends continuously on θ, the set $\mathbb{Q}_N$ is open. The intersection $\mathbb{Q}(\theta) \cap (\bigcup_N \mathbb{Q}_N) = \bigcap_N \mathbb{Q}_N(\theta) = P_{mix}(\theta) \times \{\theta\} = \mathbb{Q}_{mix} \cap \mathbb{Q}(\theta)$, hence, $\bigcap_N \mathbb{Q}_N = \mathbb{Q}_{mix}$ is a G_δ. Since $\mathbb{Q}_{mix} \cap \mathbb{Q}(\theta)$ is dense in $\mathbb{Q}(\theta)$ at least for irrational θ which are dense in S^1, $\mathbb{Q}_{mix}$ is dense in $\mathbb{Q}$.
□

<u>References</u>

[G] E. Gutkin, Billiards on almost integrable polyhedral surfaces, Erg. Th. and Dyn. Syst., 4, N4(1984), 569-584.

[H] P. R. Halmos, Lectures on Ergodic Theory, Tokyo, 1956.

[K1] A. Katok, Constructions in Ergodic Theory, Part 1, preprint, to appear in Birkhauser, Progress in Math.

[K2] A. Katok, Ergodicity of generic irrational billiards, Abstracts from workshop on 2-manifolds and Geometry, Oct. 1984, MSRI, Jan. 1986.

[K3] A. Katok, The growth rate for the number of singular and periodic orbits for a polygonal billiard, preprint, 1984.

[K4] A. Katok, Interval exchange transformations and some special flows are not mixing, Israel J. of Math. 35 (1980), 301-310.

[KMS] S. Kerckhoff, H. Masur, J. Smillie, Ergodicity of billiard lows and quadratic differentials, to appear in Ann. of Math.

[S] E. A. Sataev, On the number of invariant measures for flows on orientable surfaces, Math. of the USSR, Izrestija 9(1975), 813-830.

[W] P. Walters, An introduction to ergodic theory, Springer-Verlag, 1982.

[ZK] A Zemlyakov, A. Katok, Topological transitivity of billiards in polygons, Math. Notes of the USSR Acad. Sci. 18 N2(1975), 760-764.