NETS AND QUASI-ISOMETRIES

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This paper was written in the middle of 1980s and was accepted at that time by a leading mathematical journal modulo insignificant revisions. By essentially trivial reasons that did not involve either mathematical contents of the paper or any disagreement among the authors revised version was never submitted. Current publication is an initiative of the second author who takes full responsibility for any statements that may have become outdated and for possibly not citing more recent results that could be relevant to the material presented in the paper.

1. Basic definitions and extension lemma

Let X be a complete metric space with the distance function d_X.

Definition 1. A subset $\Gamma \subset X$ is called a *net* if:

(1.1) Γ is uniformly discrete, i.e. there is $r > 0$ such that for $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, $d_X(\gamma_1, \gamma_2) \geq r$

(1.2) Γ *spans* X, i.e. there exists $R > 0$ such that for every $x \in X$ one cane find $\gamma \in \Gamma$ with $d_X(x, y) < R$.

The infimum of all R satisfying (1.2) will be called the *spanning constant* for Γ.

Now let X, Y be two metric spaces.

Definition 2. A map $\varphi : X \to Y$ is called a *quasi-isometric embedding* if there exist positive numbers A, B such that for any $x_1, x_2 \in X$, $x_1 \neq x_2$:

$$A < \frac{d_Y(\varphi x_1, \varphi x_2)}{d_X(x_1, x_2)} < B$$

A one-to-one quasi-isometric embedding is called a *quasi-isometry*. It follows from the definition that the inverse map to a quasi-isometry is also a quasi-isometry.

Definition 3. [4, p. 7] A continuous map $\varphi : X \to Y$ is called a pseudo-isometry if for some positive constants A, B, C and for any $x_1, x_2 \in X$ one has:

$$Ad_X(x_1, x_2) - C < d_Y(\varphi x_1, \varphi x_2) < Bd_X(x_1, x_2) + C \tag{1.3}$$

Lemma 1. *Let Γ be a net in a metric space X, $\varphi : \Gamma \to Y$ a pseudo-isometry. Then there exists a subset $\Gamma' \subset \Gamma$ such that Γ' is also a net in X and the restriction of φ to Γ' is a quasi-isometric embedding.*

This lemma is an almost immediate corollary of the following statement.

Lemma 2. *Let Γ be a net in a complete non-compact metric space X and T be a positive number. Then there exists a subset $\Gamma' \subset \Gamma$ which is also a net in X and such that:*

(A) *For every $\gamma_1, \gamma_2 \in \Gamma'$, $\gamma_1 \neq \gamma_2$, $d_X(\gamma_1, \gamma_2) > T$.*

Proof. Let $\mathcal{B}(T)$ be the collection of all subsets of Γ satisfying (A). It is non-empty and is partially ordered by inclusion. Any ordered subset of $\mathcal{B}(T)$ has the maximal element (the union of all its elements); consequently by Zorn's lemma, there is a maximal element $\Gamma' \in \mathcal{B}(T)$. This means

* This author is currently supported by NSF grants DMS 1002554 and 1304830.

that for every point $\gamma \in \Gamma$ there is a point $\gamma' \in \Gamma'$ such that $d_X(\gamma, \gamma') < T$, because otherwise Γ' is not maximal. But then for every $x \in X$ there is a point $\gamma' \in \Gamma$ such that $d_X(x, \gamma') \leq d_X(x, \gamma) + d_X(\gamma, \gamma') \leq R + T$, i.e. Γ' is a net in X. □

Proof of Lemma 1. If X is compact, let Γ' be any one-point subset of Γ. If it is not compact, let us apply Lemma 2 with $T = \frac{2C}{A}$. Then for any $\gamma_1, \gamma_2 \in \Gamma'$ one has:

$$d_Y(\varphi\gamma_1, \varphi\gamma_2) > A d_X(\gamma_1, \gamma_2) - C \geq \frac{A}{2} d_X(\gamma_1, \gamma_2) + \left(\frac{A}{2}T - C\right) \geq \frac{A}{2} d_X(\gamma_1, \gamma_2)$$

and similarly since $B \geq A$

$$d_Y(\varphi\gamma_1, \varphi\gamma_2) < B d_X(\gamma_1, \gamma_2) + C < \frac{3}{2} B d_X(\gamma_1, \gamma_2) + \left(C - \frac{B}{2}T\right) < \frac{3}{2} B d_X(\gamma_1, \gamma_2)$$

□

Let us denote by X^n the n^{th} cartesian power of the space X with the product topology. Furthermore, let Σ_n be the standard $(n-1)$ simplex:

$$\Sigma_n = \left\{(t_1, \dots, t_n) : t_i \geq 0, i = 1, \dots, n, \sum_{i=1}^{n} t_i = 1\right\}.$$

Definition 4. A *centroid* on a metric space X is a map

$$C : \bigcup_{n=1}^{\infty} X^n \times \Sigma_n \to X$$

satisfying the following properties (1.4)-(1.7):

(1.4) For $n = 1$, $C(x, 1) = x$

(1.5) $C(x_1, \dots, x_n, t_1, \dots, t_{k-1}, 0, t_{k+1}, \dots, t_n)$
$= C(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n, t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_n)$

(1.6) C is continuous on every $X^n \times \Sigma_n$

(1.7) For every n and $R > 0$ there exists $F(R, n)$ such that if $d_X(x_i, x_j) \geq R$, $i, j = 1, \dots, n$ then $d_X(C(x_1, \dots, x_n, t_1, \dots, t_n), x_i) < F(R, n)$ for every $i = 1, \dots, n$ and $(t_1, \dots, t_n) \in \Sigma_n$.

It follows from (1.4) and (1.5) that:

$$C(x_1, \dots, x_n, 0, \dots, 0, \underset{\substack{\uparrow \\ i^{\text{th}}\text{ place}}}{1}, 0, \dots, 0) = x_i \tag{1.8}$$

Definition 5. An n-centroid on X is a map defined on $\bigcup_{k=1}^{n} X^k \times \Sigma_k$ and satisfying conditions (1.4)-(1.7).

Remark. Let us make the following identification on $\bigcup_{k=1}^{n} X^k \times \Sigma_k$: for each k and i identify $(x_1, \dots, x_k, t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_k)$ with $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k, t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_k)$ and call the corresponding topological space $\widehat{X}^n$. Then properties (1.5) and (1.6) can be expressed by saying that there is a continuous map $\widehat{C} : \widehat{X}^n \to X$ such that $C = \widehat{C} \circ \pi$, where $\pi : \bigcup_{k=1}^{n} X^k \times \Sigma_k \to \widehat{X}^n$ is the projection provided by the identification.

A centroid C is called *uniform* if the function $F(R, n)$ in (1.7) can be chosen independently of n. The following simple lemma is very useful for the construction of centroids.

Lemma 3. *Let X be a metric space in which balls of finite radius are compact. Then every 2-centroid on X can be extended to a centroid.*

Proof. We will use induction in n, namely we set for $n = 3, 4, \dots$

(1.9)
$$C(x_1,\dots,x_n,t_1,\dots,t_n) = C\left(C\left(x_1,\dots,x_{n-1},\frac{t_1}{t_1+\dots+t_{n-1}},\dots,\frac{t_{n-1}}{t_1+\dots+t_{n-1}}\right),x_n,t_1+\dots+t_{n-1},t_n\right)$$

if at least one of the numbers $t_1,\dots,t_{n-1}$ is positive and

$$C(x_1,\dots,x_n,0,0,\dots,1) = x_n. \tag{1.10}$$

We assume that conditions (1.5)-(1.7) hold for k-centroids $k = 2,\dots,n-1$. Condition (1.5) for n-centroid follows immediately from (1.4) and (1.5) for 2-centroid. If $k < n$ we have from (1.9) and (1.5) for $(n-1)$-centroid

$$\begin{aligned}
&C(x_1,\dots,x_n,t_1,\dots,0,\dots,t_n) = \\
&\quad = C\left(C\left(x_1,\dots,x_{n-1},\frac{t_1}{t_1+\dots+t_{n-1}},\dots,0,\dots,\frac{t_{n-1}}{t_1+\dots+t_{n-1}}\right),x_n,t_1+\dots+t_{n-1},t_n\right)\\
&\quad = C\left(C\left(x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n-1},\frac{t_1}{t_1+\dots+t_{n-1}},\frac{t_{k-1}}{t_1+\dots+t_{n-1}},\right.\right.\\
&\qquad\qquad \left.\left.\frac{t_{k+1}}{t_1+\dots+t_{n-1}},\dots,\frac{t_{n-1}}{t_1+\dots+t_{n-1}}\right),x_n,t_1+\dots+t_{n-1},t_n\right)\\
&\quad = C(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n,t_1,\dots,t_{k-1},t_{k+1},\dots,t_n)
\end{aligned}$$

Continuity of n-centroid at every point except of $(x_1,\dots,x_n,0,\dots,0,1)$ follows directly from (1.9) and from the continuity of $(n-1)$ and 2-centroids. In order to prove the continuity at $(x_1,\dots,x_n,0,\dots,0,1)$, let us notice that $(n-1)$-centroid maps a set $U \times \sigma_{n-1}$ where U is a neighborhood of $(x_1,\dots,x_n)$ with compact closure into a compact closure into a compact set A. Since 2-centroid is uniformly continuous on compact sets we have uniformly for $x \in A$ $\lim_{\substack{x'\to x_n\\ t\to 0}} C(x,x',t,t') = C(x,x_n,0,1) = x_n$.

In order to prove (1.7) for n-centroid, let us assume that we have found a function $F(R,n-1)$. We can also assume $F(R,2)$ and $F(R,n-1)$ are non-decreasing. We then have:

$$\begin{aligned} d(C(x_1,\dots,x_{n-1},t_1,\dots,t_{n-1}),x_n) &\le d_X(x_n,x_1) + \\ d_X(C(x_1,\dots,x_{n-1},t_1,\dots,t_{n-1}),x_1) &\le R + F(R,n-1)\end{aligned} \tag{1.11}$$

and from 1.9 for $i = 1,\dots,n-1$

$$\begin{aligned}
&d_X(C(x_1,\dots,x_n,t_1,\dots,t_n),x_i) \le \\
&\quad \le d_X(C(x_1,\dots,x_n,t_1,\dots,t_n),C(x_1,\dots,x_{n-1},t_1,\dots,t_{n-1})) + \\
&d_X(C(x_1,\dots,x_{n-1},t_1,\dots,t_{n-1}),x_i)
\end{aligned}$$

Using (1.7) for 2-centroid and $(n-1)$-centroid and 1.11 we see that the first term is estimated from above by $F(F(R,n-1)+R,2)$ and the second by $F(R,n-1)$. By the same reason

$$d_X(C(x_1,\dots,x_n,t_1,\dots,t_n),x_n) \le F(F(R,n-1)+R,2).$$

Thus we can put

$$F(R,n) = F(F(R,n-1)+R,2) + F(R,n-1) \tag{1.12}$$

and this function is non-decreasing □

Definition 6. We will call a metric space X *uniformly locally compact* if for any r, R the maximal number of points in any R-ball in X with pairwise distances $\geq r$ is bounded by a number depending only on R and r.

Lemma 4 (Extension Lemma)**.** *Let us assume that a metric space X is uniformly locally compact and a space Y admits a centroid. Then every pseudo-isometry $\varphi : \Gamma \to Y$ of a net $\Gamma \subset X$ into Y can be extended to a pseudo-isometry $\widetilde{\varphi} : X \to Y$ with the same constants A and B and probably different C*

Proof. Let α be a continuous non-negative function defined on the set of all positive numbers such that:

$$\alpha(t) \to \infty \quad \text{as} \quad t \to 0 \tag{1.13}$$

$$\alpha(t) > 0 \quad \text{for} \quad 0 < t \leq R \text{ where } R \text{ comes from (1.2)} \tag{1.14}$$

$$\alpha(t) = 0 \quad \text{for} \quad \text{all sufficiently large } t, \text{ say for } t \geq T. \tag{1.15}$$

Since X is uniformly locally compact the net Γ is at most countable. Actually it is infinite countable unless X is compact in which case all our considerations become trivial. Let $\gamma_1, \gamma_2, \gamma_3, \dots$ be an ordering of the elements of Γ. Let for $x \in X$

$$\Phi_\alpha(x) = \{\gamma \in \Gamma : \alpha(d_X(x,\gamma)) > 0\} \cup (\Gamma \cap \{x\})\,.$$

By (1.2) and (1.14) the set $\Phi_\alpha(x)$ is non-empty for every x. Since X is uniformly locally compact the number of elements in $\Phi_\alpha(x)$ (which we denote by $k(x)$) is bounded from above by a number K. We can represent $\Phi_\alpha(x)$ in the following form $\left\{\gamma_{i_1(x)}, \gamma_{i_2(x)}, \dots, \gamma_{i_{k(x)}(x)}\right\}$, where $i_1(x) < i_2(x) < \dots < i_{k(x)}(x)$.

Let us denote for $j = 1, \dots, k(x)$

$$\varphi\left(\gamma_{i_j}(x)\right) = \widetilde{\varphi}_j(x) \text{ and} \tag{1.16}$$

$$w_j(x) = \begin{cases} \dfrac{\alpha(d_X(x,\gamma_{i_j(x)}))}{\sum_{l=1}^{k(x)} \alpha(d_X(x,\gamma_{i_l}(x))}, & \text{if } x \notin \Gamma \\[2ex] 0, & \text{if } x \in \Gamma, \gamma_{i_j(x)} \neq x \\[2ex] 1, & \text{if } x \in \Gamma, \gamma_{i_j(x)} = x \end{cases} \tag{1.17}$$

It follows from (1.3) and (1.15) that all the points $\widetilde{\varphi}_j(x)$ lie within at most $2(BT+C)$ from each other.

We set now

$$\widetilde{\varphi}(x) = C(\widetilde{\varphi}_1(x), \dots, \widetilde{\varphi}_{k(x)}(x), w_1(x), \dots, w_{k(x)}(x)). \tag{1.18}$$

The continuity of $\widetilde{\varphi}$ follows directly from (1.16), (1.17) and the properties of the centroid. Note that (1.8) guarantees that $\widetilde{\varphi}$ is an extension of φ, and (1.1) and (1.13) provide continuity of $\widetilde{\varphi}$ at the points of Γ.

In order to check (1.3) let us take arbitrary two points $x_1, x_2 \in X$ and find points $\gamma_1, \gamma_2 \in \Gamma$ such that $d_X(x_i, \gamma_i) < R$, $i = 1, 2$. Then all the points $\widetilde{\varphi}_j(x_i)$, $j = 1, \dots, k(x_i)$ lie within the distance $2(BT+C)$ from $\varphi(\gamma_i)$ and by (1.7) and (1.18) we have

$$d_\gamma(\widetilde{\varphi}(x_i), \varphi(\gamma_i)) \leq F(2(BT+C), K)$$

and furthermore

$$\begin{aligned} d_\gamma(\widetilde{\varphi}(x_1),\widetilde{\varphi}(x_2)) &\leq d_\gamma(\widetilde{\varphi}(x_1),\varphi(\gamma_1)) + d_\gamma(\widetilde{\varphi}(x_2),\varphi(\gamma_2)) + d_\gamma(\varphi(\gamma_1),\varphi(\gamma_2)) \\ &< Bd_X(\gamma_1,\gamma_2) + C + 2F(2(BT+C),K) \\ &\leq B(d_X(x_1,x_2)+2R) + C + 2F(2(BT+C),K) \\ &= Bd_X(x_1,x_2) + C' \end{aligned}$$

where $C' = 2BR+2F((2BT+C),K)+C$. Similarly, we obtain $d_\gamma(\widetilde{\varphi}(x_1),\widetilde{\varphi}(x_2)) \geq Ad_X(x_1,x_2) - (2AR + C + 2F(2(BT+C),K) \geq Ad_X(x_1,x_2) - C'$ □

2. Examples of Centroid

According to Lemma 3 in order to construct a centroid on a metric space N it is enough to define a continuous map $C : N \times N \times [0,1] \to N$ such that

$$C(x_1,x_2,1) = x_1, \qquad C(x_1,x_2,0) = x_2 \tag{2.1}$$

and $d_N(x_1,x_2) < R$ implies

$$d_N(x_i, C(x_1,x_2,t)) < F(R) \tag{2.2}$$

We will show how to construct such a map in several important cases.

Example 1. Let N be a complete Riemannian manifold such that every two of its points are connected by a unique geodesic.

Let us denote for $x_1, x_2 \in N$ by G_{x_1,x_2} the geodesic connecting x_1 with x_2 provided with the length parameter. Since the length of this geodesic is equal to the distance $d_N(x_1,x_2)$ we can represent G_{x_1,x_2} as a map

$$G_{x_1,x_2} : [0, d_N(x_1,x_2)] \to N$$

where

$$G_{x_1,x_2}(0) = x_1 \text{ and } G_{x_1,x_2}(d_N(x_1,x_2)) = x_2. \tag{2.3}$$

Let us define a centroid $C : N \times N \times [0,1] \to N$ by

$$C(x_1,x_2,t) = G_{x_1,x_2}\left(td_N(x_1,x_2)\right) \tag{2.4}$$

Property (2.1) immediately follows from (2.3); (2.2) with $F(R) = R$ folds because

$$d_N\left(x_1, G_{x_1,x_2}(t)\right) + d_N\left(x_2, G_{x_1,x_2}(t)\right) = d_N(x_1,x_2)$$

Thus, it is left to prove the continuity of C. It is obviously continuous for $x_1 = x_2$. So let us assume that $x_1 \neq x_2$, $x_1^{(n)} \to x_1$, $x_2^{(n)} \to x_2$, $t_n \to t$ and $G_{x_1^{(n)},x_2^{(n)}}(t_n)$ does not converge to $G_{x_1,x_2}(t)$. Since all the curves $G_{x_1^{(n)},x_2^{(n)}}$ lie in a compact part of N which can be covered by a fixed number of coordinate charts one can use usual compactness argument in functional spaces to show that there is a subsequence of the sequence $G_{x_1^{(n)},x_2^{(n)}}$ which converges uniformly to a Lipschitz curve $K : [0, d_N(x_1,x_2)] \to N$ different from G_{x_1,x_2}. It is easy to see that $K(0) = x_1$, $K(d_N(x_1,x_2)) = x_2$. $d_N(x_1, K(t)) = t$ so that the length of K is equal to $d_N(x_1,x_2)$. Thus, K must be a geodesic and by uniqueness $K = G_{x_1,x_2}$.

Let us point out two particular cases to which the above construction applies.

Example 1A. N is the universal covering of a compact Riemannian manifold of non-positive sectional curvature

Example 1B. $N = G/K$ where G is a connected semisimple Lie group, K its maximal compact subgroup. Any left-invariant metric on G which is also right-invariant with respect ot K projects into a G-left invariant on N.

Example 2. Let N be a connected Lie group of exponential type, i.e. the map $\exp : \mathfrak{n} \to N$ from the lie algebra of N into N is one-to-one, provided with a left-invariant Riemannian metric. In this case for every $x \in N$ there is exactly one one-parameter subgroup $\left\{g_t^{(x)}\right\}$ of N such that $x = g_1(x)$. Let us define

$$C(x_1, x_2, t) = x_1 g_t(x_1^{-1} x_2) \tag{2.5}$$

Since for every $x \in N$, $g_0(x) = e$, $g_1(x) = x$, condition (2.1) follows immediately from (2.5). Continuity is also obvious in this case because $g_t(x)$ depends continuously on both x and t. To verify (2.2) let us remark that since both the Riemannian metric on N and the centroid C are left-invariant one has

$$C(x_1, x_2, t) = x_1 C(e, x_1^{-1} x_2, t)$$

where $d_N(e, x_1^{-1} x_2) = d_N(x_1, x_2)$ and

$$d_N(x_1, C(x_1, x_2, t)) = d_N(e, C(e, x^{-1} x_2, t)), \tag{2.6}$$
$$d_N(x_2, C(x_1, x_2, t)) = d_N(x_1^{-1} x_2, C(e, x_1^{-1} x_2, t)). \tag{2.7}$$

But if $d_N(x_1, x_2) < R$, all elements present in the right hand parts of (2.6) and (2.7) lie in the image of an R-ball in $\mathfrak{n}$ under the exponential map. This image is a compact set and consequently is contained in a d_N ball about e. We can choose the radius of that ball as $F(R)$.

Example 3. Examples 1B and 2 can be generalized in the following way.

Let N be a metric space which is homeomorphic to Euclidean space $\mathbb{R}^m$ and which has a transitive group of isometries. Then N can be represented as $I(N)/I_0$ where $I(N)$ is the connected component of the identity in the group of isometries of N and I_0 is the stabilizer of a point x_0 in $I(N)$. $I(N)$ is a Lie group and I_0 is its Lie subgroup so that $I(N)$ is a locally trivial fibered bundle over N. The fiber over a point $x \in N$ consists of all isometries from $I(N)$ which map x into x_0. Since N is contractible this fibered bundle is trivial, i.e. there is a continuous section $\psi : N \to I(N)$. Obviously, for $x \in N$

$$\psi(x) \cdot x = x_0 \tag{2.8}$$

Let us fix a homeomorphism $\Phi : \mathbb{R}^m \to N$ which maps the origin into x_0. Let for $x \in N$

$$\Phi^{-1}(x) = (s_1(x), \dots, s_m(x))$$

and for $t \in \mathbb{R}$

$$g_t(x) = \Phi(t s_1(x), \dots, t s_m(x))$$

Obviously

$$g_0(x) = x_0, \qquad g_1(x) = x \tag{2.9}$$

We are now ready to define a centroid:

$$C(x_1, x_2, t) = (\psi(x))^{-1} (g_{1-t}(\psi(x_1) x_2)) \tag{2.10}$$

Condition (2.1) follows directly from (2.10), (2.9) and (2.8). For,

$$C(x_1, x_2, 1) = (\psi(x_1))^{-1} (g_0(\psi(x_1) x_2)) = (\psi(x_1))^{-1} x_0 = x_1$$

and similarly

$$C(x_1, x_2, 0) = (\psi(x_1))^{-1}(g_1(\psi(x_1)x_2)) = (\psi(x_1))^{-1}\psi(x_1)x_2 = x_2$$

Continuity follows from the continuity of the section ψ and from (2.10). Finally, (2.2) can be proved as in Example 2. Namely,

$$C(x_1, x_2, t) = (\psi(x_1))^{-1}C(x_0, \psi(x_1)x_2)$$

and since $\psi(x_1)$ is an isometry

$$\begin{aligned} d_N(x_0, \psi(x_1)x_2) &= d_N(x_1, x_2) \\ d_N(x_1, C(x_1, x_2, t)) &= d(x_0, C(x_0, \psi(x_1)x_2, t)) \\ d_N(x_2, C(x_1, x_2, t)) &= d(\psi(x_1)x_2, C(x_0, \psi(x_1)x_2, t)) \end{aligned}$$

But since Φ is a homeomorphism, the preimage of the R-ball around x_0 in d_N-metric is contained in a Euclidean ball of some radius, say $T(R)$; the point $C(x_0, \psi(x_1)x_2, t)$ belongs to the image of that ball which is compact a consequently is contained in some ball in d_N-metric. Let us denote the radius of this last ball by $F(R)$.

The most important particular case of the situation described above appears in the following context:

Example 3A. Let G be a connected Lie group, $K \subset G$ - a maximal compact subgroup $N = G/K$ - the homogeneous space provided with a metric invariant with respect to the left action of G. Then N is homeomorphic to a Euclidean space and consequently it admits a centroid.

3. Homotopy argument

In this section we find conditions which guarantee that in the situation described in Example 3, the extension described in Lemma 4 is a surjective map.

Theorem 1. *Let M, N be two complete metric spaces homeomorphic to $\mathbb{R}^m$ and $\mathbb{R}^n$ correspondingly. Let us assume that M has a transitive group of isometries. Let, furthermore, $\varphi : M \to N$ be a continuous map such that for every $x, y \in M$*

$$d_N(\varphi(x), \varphi(y)) \geq A d_M(x, y) - C$$

for some constants A, C. Then

(3.1) $n \geq m$

(3.2) *If $m = n$, then $\varphi(M) = N$*

Proof. Let us fix a point $x_0 \in M$, a homeomorphism $\Phi : M \to \mathbb{R}^m$ which maps x_0 to the origin and a continuous section $\psi : M \to I(M)$ of the fibered bundle $I(M) \to M$.

Let us consider in the Cartesian square $M \times M$ the "thickened diagonals" with respect to d_M

$$\delta_R = \{(x, y) \in M \times M : d_M(x, y) \leq R\}$$

and also Euclidean "thickened diagonals"

$$\Delta_R = \{(x, y) \in M \times M : \Phi(\psi(x)y) \in B_R\}$$

where B_R is the Euclidean R-ball in $\mathbb{R}^m$ around the origin.

Since Φ is a homeomorphism and ψ's are isometries one can easily see that for every $R > 0$ one can find $f(R)$ such that

$$\Delta_R \subset \delta_{f(R)} \tag{3.3}$$

and

$$\delta_R \subset \Delta_{f(R)} \tag{3.4}$$

The set $M_R = (M \times M) \setminus \Delta_R$ for every R is a deformation retract of $(M \times M) \setminus \Delta = \{(x,y) : x \neq y\}$ via $(x,y) \to x$, $\psi(x)^{-1}\Phi^{-1}(t\Phi\psi(x)y))$ and thus M_R is homotopically equivalent to $(M \times M) \setminus \Delta$. Furthermore, the latter set is homotopically equivalent to the sphere $S^{m-1} = \left\{(x_1, \dots, x_m) \in \mathbb{R}^m : \sum_{i=1}^m x_i^1 = 1\right\}$ via the map

$$\nu_M : (x,y) \mapsto \frac{\Phi(x) - \Phi(y)}{||\Phi(x) - \Phi(y)||}. \tag{3.5}$$

Let $\sigma_M : M \times M \to M \times M$ be the standard involution given $\sigma_M(x,y) = (y,x)$. Obviously, $\nu_M(\sigma_M z) = -\nu_M(z)$.

Lemma 5. *Given any number R there exists a continuous map $\mu : S^{m-1} \to M_R$ such that*

$$\mu(-x) = \sigma_M(\mu x)$$

Proof. Since $\sigma_M \delta_R = \delta_R$, it follows from (3.3) and (3.4) that one can find positive numbers $R_0 = R < R_1 < \cdots < R_m$ such that

$$\sigma_M \Delta_{R_i} \subset \Delta_{R_{i+1}} \quad i = 0, \dots, m-1. \tag{3.6}$$

Let us take a point $z \in M_{R_m}$ so that by (3.6) $\sigma_M z \in M_{R_{m-1}}$. Since the set $M_{R_{m-1}}$ is homotopically equivalent to S^{m-1} it is pathwise connected so that we can connect z and $\sigma_M z$ in $M_{R_{m-1}}$ by a path λ_0. By (3.6) $\sigma_M \lambda_0 \subset M_{m-2}$. Clearly, the union $\lambda_0 \cup \sigma_M \lambda_0$ defines a continuous map

$$\mu_1 : S^1 \to M_{R_{m-2}}$$

which is skew-symmetric, i.e.

$$\mu_1(-x) = \sigma_M \mu_1(x).$$

Continuing by induction in dimension and using (3.6) we can construct maps $\mu_i : S^i \to M_{R_{m-i-1}}$ for $i = 1, \dots, m-1$ such that

$$\mu_i(-x) = \sigma_M \mu_i(x)$$

For, assuming that μ_{i-1} has been constructed and using the fact that $\pi_{i-1}(M_{R_{M-i}}) = 0$, we can extend μ_{i-1} to a continuous map

$$\lambda_{i-1} : S^i_+ \to M_{R_{m-i}}$$

where $S^i_+ = \left\{(x_1, \dots, x_{i+1}) \in S^i : x_{i+1} \geq 0\right\}$. Furthermore, since by (3.6) $\sigma_M(\lambda_{i-1}(M_{R_{m-i}})) \subset M_{R_{m-i}}$ we can extend λ_{i-1} to a skew-symmetric continuous map $\mu_i : S^i \to M_{R_{m-i-1}}$ by setting

$$\mu_i(-x) = \begin{cases} \lambda_{i-1}(x) \text{ if } x \in S^i_+ \\ \sigma_M \lambda_{i-1}(-x) \text{ if } x \in S^i \setminus S^i_+ \end{cases}$$

Finally we can put $\mu = \mu_{m-1}$. □

If the number R in Lemma 5 is chosen sufficiently large, e.g. $R > f\left(\frac{2C}{A}\right)$, where the function f comes from (3.3) and (3.4), then

$$(\varphi \times \varphi) M_R \subset (N \times N) \setminus \Delta$$

Consequently we can construct the composition map:

$$S^{m-1} \xrightarrow{\mu} M_R \xrightarrow{\varphi\times\varphi} (N\times N)\setminus\Delta \xrightarrow{\nu_N} S^{n-1} \tag{3.7}$$

where ν_N is defined similarly to (3.5).

This composition map $\chi = \mu(\varphi\times\varphi)\nu_N : S^{m-1} \to S^{n-1}$ is skew-symmetric i.e. $\chi(-x) = -\chi(x)$.

By the Borsuk-Ulam theorem [1, p.337] such a map may exist only if $m \le n$ and if $m = n$ it is homotopically non-trivial. Consequently $(\varphi\times\varphi)_R = \varphi\times\varphi|_{M_R} : M_R \to (N\times N)\setminus\Delta$ is also homotopically non-trivial.

Lemma 6. *Given a point $x_0 \in M$ there is a homotopy $\eta_t : M_R \to M_R$ such that $\eta_0 = \mathrm{id}$, $\eta_1(M_R) \subset M\times\{x_0\}$.*

Proof. Let $\alpha_t : M \to M$ be the following standard homotopy between the identity and the map which sends all M to x_0:

$$\alpha_t(x) = \Phi^{-1}((1-t)\Phi(x)) \qquad\qquad 0\le t\le 1$$

Then let

$$\eta_t(x,y) = \big(\alpha_t x, \psi(\alpha_t x)^{-1}\psi(x)y\big)$$

Obviously, $\eta_0 = \mathrm{id}$ and $\eta_1(x,y) = \big(\alpha_1 x, \psi(\alpha_1 x)^{-1}\psi(x)y\big) = \big(x_0, \psi(x_0)^{-1}\psi(x)y\big) = (x_0, \psi(x)y) \in \{x_0\}\times M_{R,x_0}$ where

$$M_{R,x_0} = M\setminus\Phi^{-1}(B_R)$$

□

We can include the restriction φ to M_{R,x_0} into the following commutative diagram

$$\begin{array}{ccc} M_{R,x_0} & \xrightarrow{\varphi} & N\setminus\{\varphi(x_0)\} \\ \Big\downarrow{\scriptstyle *\times\mathrm{id}} & & \Big\downarrow{\scriptstyle *\times\mathrm{id}} \\ M_R & \xrightarrow{(\varphi\times\varphi)_R} & (N\times N)\setminus\Delta \end{array}$$

where $*\times\mathrm{id}$ is the standard embedding: $(*\times\mathrm{id})(x) = (x_0, x)$ which is a homotopy equivalence according to Lemma 6. Thus, since $(\varphi\times\varphi)_R$ is homotopically non-trivial, the map $\varphi : M_{R,x_0} \to N\setminus\{\varphi(x_0)\}$ has the same property.

For every $x \in M_{R,x_0}$ we have from the assumption of the theorem $d_N(\varphi(x),\varphi(x_0)) > \mathrm{Ad}_M(x_1,x_2) - C > AR - C$.

In particular, if we fix a point $y_0 \in N$ in advance, we can then choose R big enough so that $y_0 \notin \varphi(M_{R,x_0})$ and, moreover, there exists a homotopy $\beta_t : N \to N$ identical on $\varphi(M_{R,x_0})$ and such that $\beta_0 = \mathrm{id}$ and $\beta_1(\varphi(x_0)) = y_0$.

This implies that $\varphi : M_{R,x_0} \to N\setminus\{y_0\}$ is also nontrivial. On the other hand, since M is contractible, the map

$$\varphi : M_{R,x_0} \to \varphi(M) \subset N$$

is homotopically trivial which means that $\varphi(M)$ can not miss the point y_0.

□

4. Main Results

Theorem 2. *Let $M = G/K$, $N = H/L$ be the factors of connected Lie groups G, H by their maximal compact subgroups K and L. Let us fix on M and N a G and H-invariant Riemannian metrics correspondingly. Let $\Gamma \subset M$ be a net with the spanning constant R and $\varphi : \Gamma \to N$ be a pseudo-isometry. Then*

(4.1) $\dim N \ge \dim M$

(4.2) *φ can be extended to a continuous pseudo-isometry $\widetilde{\varphi} : M \to N$*
(4.3) *If $\dim N = \dim M$, then $\widetilde{\varphi}(M) = N$ and the set $\varphi(\Gamma)$ spans N with the spanning constant $R_1 \le BR + C$ where the constants B, C are determined by the pseudo-isometry φ via (1.3).*

Proof. Since the space M admits a transitive group of isometries it is uniformly locally compact (cf. Definition 6). The space N admits a centroid (cf. Example 3A). Thus, we can apply Lemma 4 and extend φ to a continuous pseudo-isometry $\widetilde{\varphi} : M \to N$. All assumptions of Theorem 1 hold for the map $\widetilde{\varphi}$. Thus $\dim N \ge \dim M$ and if $\dim N = \dim M$ then $\widetilde{\varphi}(M) = N$ there exists $\gamma \in \Gamma$ such that $d_M(x, \gamma) < R$. Consequently by (1.3) one has

$$d_N(\varphi(\gamma), y) = d_N(\varphi(\gamma), \widetilde{\varphi}(x)) \le B d_M(x, \gamma) + C < BR + C$$

□

Theorem 3. *Let Γ be a net in a connected Lie group G provided with a left-invariant Riemannian metric*

(a) If $\varphi : \Gamma \to G$ is a pseudo-isometry, then $\varphi(\Gamma)$ spans G.
(b) If $\varphi : \Gamma \to G$ is a quasi-isometric embedding then $\varphi(\Gamma)$ is a net in G.

In both cases (a) and (b) the spanning constant for $\varphi(\Gamma)$ is determined yb G, the spanning constant for Γ and the constants B and C from (1.3).

Proof. If φ is a quasi-isometric embedding, then $\varphi(\Gamma)$ is obviously uniformly discrete. Thus, it is enough to prove (a).

Let K be a maximal compact subgroup of G, $\pi : G \to N = G/K$ be the standard projection. Since all left-invariant metric on G are equivalent, we can assume that the chosen metric d_G is a two-sided K-invariant so it generates a metric d_N on N invariant with respect to the left action of G. Let D be the diameter of K in G. Then we have for $g_1, g_2 \in G$

$$d_G(g_1, g_2) - D \le d_N(\pi g_1, \pi g_2) \le d_G(g_1, g_2) \tag{4.4}$$

These inequalities implie that both π and $\pi \circ \varphi : \Gamma \to N$ are pseudo-isometries. By Lemma 1 there exists a next $\Gamma' \subset \Gamma$ in G such that the restriction of π to Γ' is a quasi-isometric embedding. By the same lemma there is another net $\Gamma'' \subset \Gamma'$ such that $\pi \circ \varphi|_{\Gamma''} : \Gamma'' \to N$ is a quasi-isometric embedding. Consequently

$$\pi \varphi \pi^{-1} : \pi(\Gamma'') \to N$$

is also a quasi-isometric embedding, its image being $\pi(\varphi(\Gamma''))$. Theorem 2 says then that $\pi(\varphi(\Gamma''))$ spans N and by (4.4) $\varphi(\Gamma'')$ and hence $\varphi(\Gamma)$, spans G.

□

5. Applications to Ergodic Theory

Let (X, μ) be a Lebesgue measure space, i.e. a separable complete non-atomic probability measure space and let $\delta = \{S_\gamma\}_{\gamma \in \Gamma}$ be a measurable right action of a locally compact second countable group Γ on X be non-singular transformations. Let G be another locally compact second countable group. A measurable function $\alpha : X \times \Gamma \to G$ is called a G=cocycle over the action g if for a.e. $x \in X$ and for every $\gamma_1, \gamma_2 \in \Gamma$.

$$\alpha(x, \gamma_2 \gamma_1) = \alpha(x, \gamma_1) \alpha(\delta_{\gamma_1} x, \gamma_2) \tag{5.1}$$

The construction of Mackey range [5], [3] Section 8, allows us to associate with any G cocycle α over a right Γ action a left action of G. Namely we first determine a G-extension $g^\alpha = \left\{S_\gamma^\alpha\right\}_{\gamma \in \Gamma}$ of g which acts on $X \times G$ by

$$S_\gamma^\alpha(x, g) = (\delta_\gamma x, g\alpha(x, \gamma)) \tag{5.2}$$

It is easy to see that the cocycle equation (5.1) is equivalent to the group property for that extension $S^\alpha_{\gamma_1} S^\alpha_{\gamma_2} = S^\alpha_{\gamma_2\gamma_1}$.

The group G acts on $X \times G$ by the left shifts $L_{g_0} : L_{g_0}(x, g) = (x, g_0 g)$ and this action obviously commutes with the extension S^α. In particular, this action maps orbits of S^α into orbits and thus we can consider the factor action of G in the space of orbits of S^α. This action is called Mackey range of α and is denoted by $\mathcal{L}^\alpha$. In general the space of S^α orbits may not have good measurable structure and event if it has such a structure and S is measure preserving, the natural $\mathcal{L}^\alpha$ invariant measure may be infinite. We will give a sufficient condition which guarantees that the factor space has a structure of Lebesgue space with a natural finite invariant measure.

Let us assume that Γ is finitely generated discrete group and that G is a locally compact Lie group. The word-length metrics on Γ determined by different systems of generators are equivalent in the sense that the identity map is a quasi-isometry. Similarly, all left-invariant Riemmanian metrics on G are equivalent. Thus the notions of quasi-isometric embedding and pseudo-isometry from Γ to G are intrinsically defined.

Any left-invariant metric on Γ can be transferred to any orbit of a right Γ action. Thus, in our case the orbits of the action S are provided with a natural class of metrics defined up to a quasi-isometry.

Definition 7. A G-cocycle α over a Γ action S is called a Lipschitz cocycle if for almost every $x \in X$ the map $\alpha_x : \Gamma \to G$, $\alpha_x(\gamma) = \alpha(x, \gamma)$ is a pseudo-isometry with constants A, B, C (cf. (1.3)) independent on x.

Theorem 4. *Let Γ be a uniform lattice in a connected Lie group G and let α be a Lipschitz G cocycle over a right measurable non-singular action S of Γ on a Lebesgue space (X, μ). Then the Γ-action S^α has a measurable fundamental domain $D = \bigcup_{x\in X} \{x\} \times D_x$ where all sets D_x are uniformuly bounded and each D_x has a boundary of codimension one. Consequently if the action S preserves the measures μ and if the group G is unimodular, the restriction of the measure $\mu \times \chi_G$ (χ_G is the Haar measure of G) to D determines a finite invariant measure for the Mackey range $\mathcal{L}^\alpha$.*

Proof. It follows from the cocycle equation (5.1) that $\alpha(x, \mathrm{id}_\Gamma) = \mathrm{id}_G$ and

$$\alpha(x, \gamma^{-1}) = \alpha\left(S_{\gamma^{-1}}, \gamma\right)^{-1} \tag{5.3}$$

By Theorem 3 the set $\alpha_x(\Gamma)$ for almost every $x \in X$ spans G with a uniformly bounded spanning constant with respect to any left invariant Riemannian metric on G. The inversion $g \mapsto g^{-1}$ maps left-invariant metrics on G into right invariant ones. From now on we will work with a fixed right-invariant Riemannian metric d_G on G. Thus, by the above remark for almost every $x \in X$, the set $(\alpha_x(\Gamma))^{-1}$ spans G with respect to d_G.

Let $D_x(\gamma)$ be the Dirichlet region of the point $(\alpha_x(\gamma))^{-1}$, i.e.

$$D_x(\gamma) = \left\{g \in G : d_G\left(g, \alpha(x,\gamma)^{-1}\right) \le d_G\left(g, \alpha(x,\gamma')^{-1}\right) \text{ for all } \gamma \in \Gamma\right\}$$

Since $(\alpha_x(\Gamma))^{-1}$ is a discrete set, the boundary of each set $D_x(\gamma)$ has codimension one. Theorem 3 guarantees that all sets $D_x(\gamma)$ are compact and have bounded diameters.

We obtain, using (5.1), (5.3) and the invariance of d_G with respect to right multiplication on G, that $D_x(\gamma) \cdot \alpha(x, \gamma)$ is exactly:

$$\begin{aligned}
& \left\{g \in G : d_G\left(g(\alpha(x,\gamma))^{-1}, \alpha(x,\gamma)^{-1}\right) \le d_G\left((g\alpha(x,\gamma)^{-1}, (\alpha(x,\gamma')^{-1}\right) \text{ for all } \gamma' \in \Gamma\right\} \\
= \; & \left\{g \in G : d_G(g, \mathrm{id}) \le d_G(g, \alpha(x,\gamma')^{-1}\alpha(x,\gamma) \text{ for all } \gamma' \in \Gamma\right\} \\
= \; & \left\{g \in G : d_G(g, \mathrm{id}) \le d_G(g, \alpha(S_\gamma x, \gamma'\gamma^{-1}))^{-1} \text{ for all } \gamma' \in \Gamma\right\} \\
= \; & D_{S_\gamma x}(\mathrm{id})
\end{aligned}$$

and by (5.2)

$$S^{\alpha}_{\gamma}(\{x\} \times D_x(\gamma)) = \{S_\gamma x\} \times D_{S_\gamma x}(\mathrm{id})$$

Thus, every orbit of S^α visits the set

$$D := \bigcup_{x\in X} \{x\} \times D_x(\mathrm{id}) \tag{5.4}$$

at least once.

If we assume that for every $\gamma \neq \mathrm{id}_\Gamma$, $d(x,\gamma) \neq \mathrm{id}_G$ almost everywhere then the sets $D_x(\gamma)$ form a partition of $X \times G$ up to a set of measure zero and the set defined by (5.4) is a fundamental domain for S^α satisfying all conditions of the theorem.

In a general situation the Lipschitz condition from Definition 7 guarantees that the following equivalence relation has finite equivalence classes:

$$x \sim y \text{ if } y = S_\gamma x, \alpha(x,\gamma) = \mathrm{id}$$

Moreover, the partition of X into equivalence classes is measurable. Let us choose a measurable set $A \subset X$ which intersects each equivalence class by exactly one point and define

$$D = \bigcup_{x\in A} \{x\} \times D_x(\mathrm{id})$$

This set is obviously a fundamental domain for S^α and satisfies all conditions of the theorem.

Remark 1. Construction used in the proof of Theorem 4 is a straightforward modification of the construction from [2], Proposition 1, which deals with the case $\Gamma = \mathbb{Z}^n$, $G = \mathbb{R}^n$. Our Theorem 3 replaces the ergodic theorem and elementary index arguments in $\mathbb{R}^n$ used in [2].

Another application of our results to ergodic theory involves the extension of the notion of Kakutani equivalence of group action to various classes of non-abelian groups. The construction is described in Section 8 of [3]. Details will appear in a separate paper.

□

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