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ERRATA TO "MEASURE RIGIDITY BEYOND UNIFORM HYPERBOLICITY: INVARIANT MEASURES FOR CARTAN ACTIONS ON TORI" AND "UNIQUENESS OF LARGE INVARIANT MEASURES FOR $\mathbb{Z}^k$ ACTIONS WITH CARTAN HOMOTOPY DATA"

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In this note we correct minor errors in [2, 4] that are due to a mistake in [2, Lemma 1.2] that in turn is based on an uncritical quotation of [3, Theorem 2.6.1], which contains an error in the uniqueness statement. Lemma 1.2 from [2] is incorrect as stated (although it is true for a restriction of the action α to a subgroup of finite index) and should be replaced by Lemma 1 below. All results of [2, 4] are correct, with h being a semiconjugacy between the action α and an *affine* action α_0 with the same homotopy data as α. In [4, Corollary 2.4], "linear models" should be replaced with "affine models". The proofs are not affected.

Let us explain the nature of the error first. Let L be an integer $n \times n$ matrix with determinant ± 1 and no eigenvalues of absolute value one. It determines an automorphism F_L of the torus $\mathbb{T}^n$ that is an Anosov diffeomorphism. Let f be a diffeomorphism of $\mathbb{T}^n$ homotopic to F_L. Then there exists a continuous map $h\colon \mathbb{T}^n \to \mathbb{T}^n$ homotopic to the identity such that

$$h \circ f = F_L \circ h. \tag{1}$$

However, such a map h is in general not unique. It is defined up to a left multiplication by a transformation homotopic to the identity and commuting with F_L. Those transformations are translations by elements of the group of the fixed points of F_L.

Matrices that commute determine commuting automorphisms of the torus. Commuting diffeomorphisms homotopic to those automorphisms do not necessarily have a *common* fixed point even if the matrices are hyperbolic and those diffeomorphisms are affine maps, see [1]. Hence the abelian action generated by such diffeomorphisms is not a factor of the action by automorphisms as Lemma 1.2 in [2] erroneously states. However, the statement is true if one replaces the action by automorphisms by an action by *affine* maps, and hence is true in its original form for a finite-index subgroup of the original action. While this fact and its proof are fairly standard, we include those for the sake of completeness and to avoid the future propagation of confusion based on the string of inaccurate quotations.

LEMMA 1. *Let α be an action of $\mathbb{Z}^k$ by C^1 diffeomorphisms of $\mathbb{T}^n$ such that for some $\mathbf{m} \in \mathbb{Z}^k$, the diffeomorphism $\alpha(\mathbf{m})$ induces a hyperbolic automorphism $\alpha(\mathbf{m})_*$ on the fundamental group $\mathbb{Z}^n$. Then there exists a continuous map $h\colon \mathbb{T}^n \to \mathbb{T}^n$*

homotopic to the identity (and hence surjective) such that

$$h \circ \alpha = \alpha_0 \circ h,$$

where α_0 *is an affine action of* $\mathbb{Z}^k$ *on* $\mathbb{T}^n$ *such that for each* $\mathbf{m} \in \mathbb{Z}^k$*, the linear part of* $\alpha_0(\mathbf{n})$ *is* $\alpha(\mathbf{n})_*$.

Proof. By assumption, $\alpha(\mathbf{m})_*$ is a hyperbolic matrix in $GL(n,\mathbb{Z})$ which induces an Anosov automorphism of $\mathbb{T}^n$, we denote them respectively by A and F_A. We note that the Lefschetz number of F_A is $\pm\det(A-\mathrm{Id}) \neq 0$ (see for example [3, 8.7.1]). Since $f = \alpha(\mathbf{m})$ is homotopic to F_A, the Lefschetz number of f is also nonzero. Hence f has at least one fixed point. Conjugating α by a translation, we can assume without loss of generality that f fixes $0 \in \mathbb{T}^n$.

By a theorem of Franks, there exists a continuous map $h\colon \mathbb{T}^n \to \mathbb{T}^n$ satisfying

$$F_A \circ h = h \circ f, \quad h(0) = 0, \quad \text{and } h \text{ is homotopic to identity.} \tag{2}$$

Moreover, the map satisfying (2) is unique.[1] Let us prove uniqueness.

Suppose h' is another such map. We denote by $\tilde{f}$ the lift of f to the universal cover $\mathbb{R}^n$ that fixes 0. The corresponding lift of F_A is A and the lifts of h and h' can be written as $\tilde{h} = \mathrm{Id} + H$ and $\tilde{h}' = \mathrm{Id} + H'$, where H and H' are periodic bounded functions. Now (2) implies that for all $n \in \mathbb{Z}$, $A^n \circ \tilde{h} = \tilde{h} \circ \tilde{f}^n$, since both sides are lifts of the same map that both fix 0. Similarly, $A^n \circ \tilde{h}' = \tilde{h}' \circ \tilde{f}^n$, and hence

$$A^n \circ (\tilde{h} - \tilde{h}') = \tilde{h} \circ \tilde{f}^n - \tilde{h}' \circ \tilde{f}^n = H \circ \tilde{f}^n - H' \circ \tilde{f}^n.$$

Since H and H' are periodic bounded functions, the right-hand side is bounded uniformly in $n \in \mathbb{Z}$. Since A is a hyperbolic linear map, this forces $\tilde{h} - \tilde{h}' = 0$ identically on $\mathbb{R}^n$. This completes the proof of uniqueness.

Let us slightly abuse notation and denote the automorphism of the torus defined by the matrix $\alpha(\mathbf{n})_*$ by the same symbol. Then we can rewrite the first equation in (2) as

$$h \circ \alpha(\mathbf{m}) = \alpha(\mathbf{m})_* \circ h. \tag{3}$$

Now we consider an arbitrary $\mathbf{m}' \in \mathbb{Z}^k$. Since $\alpha(\mathbf{m}')$ commutes with $\alpha(\mathbf{m})$, $\alpha(\mathbf{m}')(0)$ is also a fixed point of $\alpha(\mathbf{m})$, and hence $q = h \circ \alpha(\mathbf{m}')(0)$ is a fixed point of $\alpha(\mathbf{m})_*$. We consider the translation $T(x) = x + q$ on $\mathbb{T}^n$ and note that T commutes with $\alpha(\mathbf{m})_*$.

Now we define $\alpha_0(\mathbf{m}')$ to be the affine map

$$\alpha_0(\mathbf{m}') = T \circ \alpha(\mathbf{m}')_*\colon \mathbb{T}^n \to \mathbb{T}^n,$$

and note that $\alpha_0(\mathbf{m}')$, and hence its inverse, commute with $\alpha(\mathbf{m})_*$. Now we consider the map

$$h' = \alpha_0(\mathbf{m}')^{-1} \circ h \circ \alpha(\mathbf{m}'). \tag{4}$$

[1] For a proof of this statement see *e.g.,* the proof of [3, Theorem 2.6.1].This is exactly what is proved there. Notice, however, that the quoted theorem states incorrectly that the map $h\colon \mathbb{T}^n \to \mathbb{T}^n$ homotopic to the identity and such that $F_A \circ h = h \circ f$ is unique, not assuming that $h(0) = 0$, and thus ignoring the possibility of a translation commuting with F_A. Uncritical quotation of the incorrect uniqueness statement of [3, Theorem 2.6.1] led to the mistake in [2, Lemma 1.2].

The definitions imply that $h'(0) = 0$ and h' is homotopic to identity. It also satisfies (3), indeed

$$
\begin{aligned}
h' \circ \alpha(\mathbf{m}) &= \alpha_0(\mathbf{m}')^{-1} \circ h \circ \alpha(\mathbf{m}') \circ \alpha(\mathbf{m}) \\
&= \alpha_0(\mathbf{m}')^{-1} \circ h \circ \alpha(\mathbf{m}) \circ \alpha(\mathbf{m}') \\
&= \alpha_0(\mathbf{m}')^{-1} \circ \alpha(\mathbf{m})_* \circ h \circ \alpha(\mathbf{m}') \\
&= \alpha(\mathbf{m})_* \circ \alpha_0(\mathbf{m}')^{-1} \circ h \circ \alpha(\mathbf{m}') \\
&= \alpha(\mathbf{m})_* \circ h'.
\end{aligned}
$$

By the uniqueness of the map satisfying (2), we conclude that $h = h'$, and hence for all $\mathbf{m}' \in \mathbb{Z}^k$ we have

$$h \circ \alpha(\mathbf{m}') = \alpha_0(\mathbf{m}') \circ h.$$

It follows that α_0 is a $\mathbb{Z}^k$ action and that α_0 and h satisfy the conclusion of the lemma. □

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