▌GAFA Geometric And Functional Analysis

THE FRIED AVERAGE ENTROPY AND SLOW ENTROPY FOR ACTIONS OF HIGHER RANK ABELIAN GROUPS

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Abstract. We consider two numerical entropy-type invariants for actions of $\mathbb{Z}^k$, invariant under a choice of generators and well-adapted for smooth actions whose individual elements have positive entropy. We concentrate on the maximal rank case, i.e. $\mathbb{Z}^k$, $k \geq 2$ actions on $k + 1$-dimensional manifolds. In this case we show that for a fixed dimension (or, equivalently, rank) each of the invariants determines the other and their values are closely related to regulators in algebraic number fields. In particular, in contrast with the classical case of $\mathbb{Z}$ actions the entropies of ergodic maximal rank actions take only countably many values. Our main result is the dichotomy that is best expressed under the assumption of weak mixing or, equivalently, no periodic factors: either both invariants vanish, or their values are bounded away from zero by universal constants. Furthermore, the lower bounds grow with dimension: for the first invariant (the Fried average entropy) exponentially, and for the second (the slow entropy) linearly.

1 Introduction

It is well-known that the standard notion of entropy for an action of a locally compact compactly generated topological group G by measure-preserving transformations assigns zero value to any smooth action unless the group G is virtually cyclic, i.e. a compact extension of $\mathbb{Z}$ (notice that, in particular, $\mathbb{R}$ is a compact extension of $\mathbb{Z}$).[1] The reason is very simple: entropy of a group action measures in some way the exponential growth of the number of distinguishable orbit segments against the size or the volume of, say, a ball in the word-length metric or some left-invariant metric in the acting group while for the smooth (or Lipschitz) actions this number growth no faster than exponentially against the radius of that ball.

There are two natural ways to obtain numerical invariants reflecting orbit growth that are non-trivial for smooth measure-preserving actions:

[1] Condition of compactly generated (finitely generated in the case of discrete groups) is essential. For example, *any* action of the direct limit of finite cyclic groups allows smooth realization (the first author, unpublished).

(1) Change the normalization and measure exponential growth against the radius of the ball. This leads to the slow entropy type invariants as in [KT97].
(2) Consider an average exponential growth, represented by the ordinary entropy, along one-parameter subgroups, or, more generally, "directions" in the group; for higher-rank abelian groups this was done by Fried [Fri83].

In this paper we only consider actions of higher rank abelian groups, more specifically, $\mathbb{Z}^k \times \mathbb{R}^n$. In fact, for the general discussion it is sufficient to consider actions of $\mathbb{R}^k$. A standard suspension construction allows to reduce actions of $\mathbb{Z}^k$, or, more generally, $\mathbb{Z}^k \times \mathbb{R}^n$ to this case. Let us fix a volume element in $\mathbb{R}^k$; in the suspension case a natural volume element comes from $\mathbb{Z}^k$. For $\mathbb{R}^k$ actions the group of automorphisms preserving the volume element and orientation is naturally $SL(k, \mathbb{R})$; for $\mathbb{Z}^k$ actions the group automorphisms is $SL(k, \mathbb{Z})$. If the orientation is not fixed, one takes corresponding two-point extensions of those groups.

A notion of entropy for $\mathbb{R}^k$ actions in order to be useful in the smooth case should be:

 (i) invariant under volume preserving automorphisms of $\mathbb{R}^k$; [2]
 (ii) equal to the usual entropy for $k = 1$;
(iii) positive if entropy of all non-identity elements of the action is positive;
(iv) finite for smooth actions on compact manifolds with respect to any Borel probability invariant measure.

Fried [Fri83] suggested a notion of entropy for actions of higher rank abelian groups satisfying these properties, based on the averaging approach (2). We discuss this notion in section 2. For $\mathbb{Z}^k$ actions one can also define entropy type invariant for any rank between 1 and k based on Fried's approach, see Definition 3.

A notion based on the slow entropy approach (1) is described in section 4.1. While in the Fried definition the invariance property (i) is automatic, in the slow entropy approach it is only achieved by minimization over a class of norms in the acting group that of course makes calculations more difficult.

Principal parts of this paper, sections 3 and 4.4, deal with a special class of actions, namely actions of $\mathbb{Z}^{n-1}$ by hyperbolic automorphisms of the torus $\mathbb{T}^n$, $n \geq 3$ (Cartan actions).

In section 3 we use sophisticated number-theoretic tools to show that the Fried average entropy (Definition 2) of such actions is bounded from below by a positive constant, obtain a good below estimate for that constant, and calculate the minimal value that is conjecturally achieved for a certain $\mathbb{Z}^3$ action on $\mathbb{T}^4$; see the first line of Table 2. Furthermore, the lower bound grows exponentially with the rank of a Cartan action, see Theorem 3.1. We also obtain lower bounds (polynomial and exponential) for many (but not all) intermediate entropies (Propositions 3.4 and 3.5). An exponential bound for the remaining case is formulated as Conjecture 2.

The reason for particular attention we pay to Cartan actions is that they and their minor modifications provide the only models of $\mathbb{Z}^{n-1}$ actions with positive

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[2] In the suspension case this amounts to invariance under a choice of generators in $\mathbb{Z}^k$.

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Fried average entropy on n-dimensional manifolds for $n \geq 3$. This is proved in [KR] and is quoted below as Theorem A. Thus our estimates of the Fried average entropy and intermediate entropies are universal for weakly mixing actions of $\mathbb{Z}^{n-1}$ on n-dimensional manifolds, $n \geq 3$ with positive Fried average entropy, see Corollary 3.1.

In section 4.4 we calculate our automorphism invariant slow entropy for Cartan actions. Our main conclusion (Theorem 4.1) is that it is uniquely determined by the dimension and the value of the Fried average entropy. Moreover, it is uniformly bounded away from zero. Again, the measure rigidity Theorem A implies that the same is true for any weakly mixing action of $\mathbb{Z}^{n-1}$ with positive slow entropy on an n-dimensional manifold for $n \geq 3$.

Dropping the weak mixing assumption that in our situation is equivalent to the absence of periodic factors, allows to construct examples with arbitrary small but positive Fried average entropy and slow entropy, albeit on manifolds of increasing complexity [KR, Remark 7].

2 Entropy Function and the Fried Average Entropy

Let α be an action of $\mathbb{R}^k$ preserving a probability ergodic measure μ. We will only consider actions all of whose elements have finite entropy as in the smooth case.

DEFINITION 1. *The* entropy function *for the action α, denoted by h_μ^α, associates to* $\mathbf{t} = (t_1, \ldots t_k) \in \mathbb{R}^k$ *the value of measure-theoretic entropy of $\alpha(\mathbf{t})$, i.e. $h_\mu^\alpha(\mathbf{t}) = h_\mu(\alpha(\mathbf{t}))$.*

For general actions the entropy function is not known to possess any nice properties other than the obvious positive homogeneity of degree one and central symmetry; see [OW87, Appendix A(b)] for a specific pathological example. However for smooth actions this function is convex and piece-wise linear [Hu93]. Lyapunov exponents of the action α for such measure μ are linear functionals $\chi_j : \mathbb{R}^k \to \mathbb{R}, 1 \leq j \leq n$, and their kernels are called the *Lyapunov hyperplanes*. We will routinely ignore k zero Lyapunov exponents that come from the orbit directions of the action. Connected components of the complement to the union of the Lyapunov hyperplanes are called *Weyl chambers*. For any $C^{1+\epsilon}$ action on a compact manifold, or, more generally, for an action on a smooth manifold which is uniformly $C^{1+\epsilon}$, the entropy function is linear inside each Weyl chamber.[3] In general, maximal cones of linearity may comprise several Weyl chambers. If maximal cones of linearity coincide with Weyl chambers we say that the action satisfies *full entropy condition* with respect to the measure μ. Full entropy condition is always satisfied if the measure μ is absolutely continuous. In this case, by the Pesin entropy formula,

$$h_\mu^\alpha(\mathbf{t}) = h_\mu(\alpha(\mathbf{t})) = \sum_{j:\chi_j>0} \chi_j(\mathbf{t}), \quad \text{for } \mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k. \quad (2.1)$$

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[3] There are other cases of piecewise linearity that, however, require dynamical assumptions; see e.g. [BL97, Theorem 6.16] where the role of Weyl chambers is played by the cones of expansive directions in a continuous $\mathbb{Z}^k$ action.

Since the measure μ is absolutely continuous, we have $\sum_{j=1}^n \chi_j = 0$, therefore (2.1) can be re-written as

$$h_\mu^\alpha(\mathbf{t}) = h_\mu(\alpha(\mathbf{t})) = \frac{1}{2}\sum_{j=1}^n |\chi_j(\mathbf{t})|. \tag{2.2}$$

Now we return to the general case of an ergodic Borel probability measure. There are two possibilities:

(**P**) *all elements of the action, save for the identity, have positive entropy;*
(**O**) *some non-trivial element of the action has zero entropy.*

In the first case the entropy function defines a norm in $\mathbb{R}^k$; the unit ball is a polyhedron whose faces are sections of maximal cones of linearity. In the second case the entropy function is a semi-norm.

DEFINITION 2. The Fried average entropy *of an action α of $\mathbb{R}^k$ with a fixed volume element, denoted by $h^*(\alpha)$, is the inverse of the volume of the unit ball $B(h_\mu^\alpha)$ in the entropy function norm/semi-norm multiplied by the volume of the generalized octahedron in $\mathbb{R}^k$, (the unit ball in the ℓ^1 norm) $\mathcal{O} = \{(x_i) \in \mathbb{R}^k | \sum_{i=1}^k |x_i| \leq 1\}$, which is equal to $\frac{2^k}{k!}$. Thus*

$$h^*(\alpha) = \frac{2^k}{k!\,vol(B(h_\mu^\alpha))}.$$

The Fried average entropy of a $\mathbb{Z}^k$ action α is defined as the Fried average entropy of its suspension.

For a mixed group $\mathbb{Z}^a \times \mathbb{R}^b$ with $a + b = k$ one takes suspensions for the discrete generators; as before the volume element in $\mathbb{R}^b$ is assumed to be fixed.

REMARK 1. Fried was aware of the polyhedral structure of the entropy function for absolutely continuous measures, but not in the general smooth case since the latter relies on the Ledrappier–Young generalization of the Pesin entropy formula [LY85] that appeared later than Fried's work, and was explicitly established even later by Hu [Hu93].

The normalization is chosen in such a way that the following properties hold:

PROPOSITION 2.1.   (1)   *Let α_1 be an action of $\mathbb{R}^{k_1}$ on M_1, α_2 be an action of $\mathbb{R}^{k_2}$ on M_2, preserving the measures μ_1 and μ_2, correspondingly, and $\alpha = \alpha_1 \times \alpha_2$ be the product action of $\mathbb{R}^{k_1+k_2}$ with the measure $\mu = \mu_1 \times \mu_2$. Then $h^*(\alpha) = h^*(\alpha_1)h^*(\alpha_2)$.*
(2)   *For $k = 1$, $h^*(\alpha)$ is equal to the usual entropy of the flow, i.e. the entropy of its time-one map.*

Proof.   (1)   Let $B(h_{\mu_1}^{\alpha_1})$ and $B(h_{\mu_2}^{\alpha_2})$ be the unit balls in the entropy norms for α_1 and α_2, respectively, and $B(h_\mu^\alpha)$—the unit ball in the entropy norm for α. We use the additivity of entropy for Cartesian products and the following property of the volume in ℓ^1-norm on a product of Euclidean spaces provided with norms:

$$(k_1 + k_2)!vol(B(h_\mu^\alpha)) = k_1!vol(B(h_{\mu_1}^{\alpha_1}))k_2!vol(B(h_{\mu_2}^{\alpha_2})).$$

Then

$$h^*(\alpha) = \frac{2^{(k_1+k_2)}}{vol(B(h_\mu^\alpha))(k_1 + k_2)!} = \frac{2^{k_1}}{k_1!vol(B(h_{\mu_1}^{\alpha_1}))}\frac{2^{k_2}}{k_2!vol(B(h_{\mu_2}^{\alpha_2}))}$$
$$= h^*(\alpha_1)h^*(\alpha_2).$$

(2)   For $k = 1$, the volume of the unit ball in the ℓ^1-norm in $\mathbb{R}$, $\mathcal{O} = [-1, 1]$ is equal to 2. The entropy function is given by $h(t) = th_\mu((\alpha(1)))$. Hence $h_\mu(\alpha) = h_\mu(\alpha(1)) = \frac{2}{vol(B(h_\alpha))} = h^*(\alpha)$. $\square$

The following properties of the Fried average entropy follow immediately from the definition:

PROPOSITION 2.2.   (1)   *The Fried average entropy is equal to zero exactly in case* (**O**).
(2)   *The Fried average entropy of a $\mathbb{Z}^k$ action is independent of the choice of generators.*
(3)   *The Fried average entropy of the restriction of a $\mathbb{Z}^k$ action α to a finite index subgroup $A \subset \mathbb{Z}^k$ is equal to h_μ^α multiplied by the index of the subgroup A.*

DEFINITION 3. *For $1 \leq l \leq k$ the l-entropy of a $\mathbb{Z}^k$ action α, denoted by h_μ^ℓ, is the infimum of Fried average entropies of $\mathbb{Z}^l$ subactions β of α:*

$$h^\ell(\alpha) = \inf_{\beta \subset \alpha} h^*(\beta).$$

In particular, k-entropy of a $\mathbb{Z}^k$ action is its Fried average entropy.

3   Algebraic Actions and Rigidity of the Fried Average Entropy for Maximal Rank Actions

3.1   Maximal rank actions.   We will call an action of $\mathbb{Z}^a \times \mathbb{R}^b$ with $a+b = n-1$ on a $n + b$-dimensional manifold a *maximal rank action*. Any action whose rank is higher than that necessarily has zero Fried average entropy since in this case intersection of all Lyapunov hyperplanes contains a non-zero element and hence suspension of the action has an element all of whose Lyapunov exponents are non-positive and hence has zero entropy.

We will say that α is a *general position action* if the the Lyapunov hyperplanes, are in general position, i.e. the intersection of any number of those has the minimal possible dimension.

Notice that if α is a general position maximal rank $\mathbb{Z}^k$ action with $k = n - 1$, the number of Weyl chambers (where all Lyapunov exponents have the same signs) is equal to $2^n - 2$. Therefore either

(1)   there is a Weyl chamber where all signs are negative, or
(2)   all sign combinations excepts for all pluses and all minuses appear in some Weyl chambers.

In the case (1) the measure has to be atomic [Kat96], hence the entropy function is identically zero. A simple argument shows that the general position assumption can be removed.

PROPOSITION 3.1. *If α is not a general position maximal rank action then its Fried average entropy is equal to zero.*

Proof. Let m be the maximal number of Lyapunov hyperplanes whose intersection has dimension greater than minimal, i.e. greater than $n - 1 - m$. Consider the restriction of the action to the intersection of those m Lyapunov hyperplanes that we denote by L. Intersections of remaining $n - m$ hyperplanes with L are in general position; hence they divide L into 2^{n-m} domains where the exponents have all possible combinations of signs. In particular there is a domain D where all $n - m$ exponents are negative. Since the remaining exponents vanish on L all action elements from D have zero entropy. $\square$

So we only consider the case (2). In that case, unless the entropy function is identically zero, the measure μ is absolutely continuous [KKR11]. To summarize, the (deep) results of [KKR11] and the above arguments provide the following dichotomy for ergodic invariant measures for maximal rank actions of rank $k \geq 2$:

($\mathfrak{D}$)*Either the measure is absolutely continuous and the Fried average entropy is positive, or Fried average entropy of the action is equal to zero.*

3.2 Cartan actions.   Examples of $\mathbb{Z}^{n-1}$ actions with positive Fried average entropy (and positive l-entropies for all l, $1 \leq l \leq n - 1$) are Cartan actions by automorphisms or affine maps of the torus $\mathbb{T}^n$ and its finite factors called the infratori (see e.g. [KN11, Section 2.1.4]) with respect to Lebesgue measure.

The entropy function for those actions is described in terms of groups of units in the rings of integers in algebraic number fields as follows, see [KKS02]. Let $A \in SL(n, \mathbb{Z})$ be a matrix with an irreducible characteristic polynomial and hence distinct eigenvalues. The centralizer of A in the ring $M(n, \mathbb{Q})$ of $n \times n$ matrices with rational coefficients can be identified with the ring of all polynomials in A with rational coefficients modulo the principal ideal generated by the polynomial $f(A)$, and hence

with the field $K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of A. This identification is given by an injective map

$$\gamma : p(A) \mapsto p(\lambda) \tag{3.1}$$

with $p \in \mathbb{Q}[x]$. Let $\mathcal{O}_K$ denote the ring of integers in K and $\mathcal{U}_K$ the group of units in $\mathcal{O}_K$. Notice that if $B = p(A) \in M(n, \mathbb{Z})$ then $\gamma(B) \in \mathcal{O}_K$, and if $B \in GL(n, \mathbb{Z})$ then $\gamma(B) \in \mathcal{U}_K$ (converse is not necessarily true). We denote the centralizer of A in $M(n, \mathbb{Z})$ by $C(A)$, and the centralizer of A in the group $GL(n, \mathbb{Z})$ by $Z(A)$. It was shown in [KKS02] that $\gamma(C(A))$ is an order in K,

$$\mathbb{Z}[\lambda] \subset \gamma(C(A)) \subset \mathcal{O}_K, \tag{3.2}$$

and $Z(A)$ is isomorphic to the group of units in $\gamma(C(A))$. Therefore, by the Dirichlet Unit Theorem [BS66], to $\mathbb{Z}^{r_1+r_2-1} \times F$, where r_1 is the number of the real embeddings, r_2 is the number of pairs of complex conjugate embeddings of the field K into $\mathbb{C}$, and F is a finite cyclic group. We have $r_1 + 2r_2 = n$, and for the Cartan action, i.e. when $r_1 + r_2 - 1 = n - 1$, we conclude that $r_2 = 0$, i.e. K is a totally real number field, and $F = \{\pm 1\}$.

Let $A_1 = A, A_2, \ldots, A_{n-1} \in Z(A)$ correspond to the multiplicatively independent units of K, $\varepsilon_1, \ldots, \varepsilon_{n-1}$. More precisely, let $v = (v_1, \ldots, v_n)$ be an eigenvector of A with eigenvalue λ whose coordinates belong to K. By (3.1) $\gamma(A) = \lambda$ is a unit of K, call it ε_1. We have $A_i = p_i(A)$, hence $A_i v = p_i(\lambda)v$, where $p_i(\lambda)$ is another unit of K, ε_i. Let $\varphi_1 = id, \varphi_2, \cdots, \varphi_n$ be different embedding of K into $\mathbb{R}$. Then by taking different embeddings of the relation $A_i v = \varepsilon_i v$, we obtain $A_i \varphi_j(v) = \varphi_j(\varepsilon_i)\varphi_j(v)$, i.e. the eigenvalues of A_i are units conjugate to ε_i, and we obtain the following result.

PROPOSITION 3.2. A_i is conjugate to

$$\begin{pmatrix} \varepsilon_i & 0 & \cdots & 0 \\ 0 & \varphi_2(\varepsilon_i) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \varphi_n(\varepsilon_i) \end{pmatrix}$$

over $\mathbb{R}$.

Any Cartan action α by commuting automorphisms $A_1, \ldots, A_{n-1}$ of the torus $\mathbb{T}^n$ is described by the above construction.

Now we will discuss the ℓ-entropies of Cartan actions for fixed values of ℓ, and the Fried average entropy.

The 1-entropy is simply the minimal value of the entropy of a non-identity element of the action. It is equal to the logarithm of the Mahler measure of the corresponding unit in the field K. By [Sch73, HS93] for totally real fields the Mahler measure of any unit grows exponentially with the exponent bounded from below by

$$c = \frac{1}{2} \log\left(\frac{1 + \sqrt{5}}{2}\right). \tag{3.3}$$

Thus we have the following statement:

PROPOSITION 3.3. *The 1-entropy of a Cartan action of a given rank $n-1$ is bounded from below by the linear function cn.*

Adding some simple geometric considerations, we obtain a more general result:

PROPOSITION 3.4. *The ℓ-entropy of a Cartan action of rank $n-1$ is bounded from below by $\frac{c^\ell n^\ell}{\ell!}$.*

Proof. Consider a $\mathbb{Z}^\ell$ subaction β of a Cartan action α. We need to estimate the Fried average entropy of β, $h^*(\beta)$. The entropy norm h_μ^α restricts to $\mathbb{R}^\ell$ as h_μ^β, and by Proposition 3.3 for each element of the lattice $\mathbf{k} \in \mathbb{Z}^\ell$ $h_\mu^\beta(\mathbf{k}) > cn$. Consider a ball in the entropy norm h_μ^β of radius cn. Since it contains no points of the lattice $\mathbb{Z}^\ell$ except for the origin, by Minkowski theorem, its volume is less then 2^ℓ. Thus

$$c^\ell n^\ell vol(B(h_\mu^\beta)) < 2^\ell,$$

hence

$$vol(B(h_\mu^\beta)) < \frac{2^\ell}{c^\ell n^\ell}.$$

Therefore

$$h^*(\beta) > \frac{c^\ell n^\ell}{\ell!}. \qquad \square$$

Using Stirling formula we see that Proposition 3.4 implies exponential below estimate for the ℓ-entropies for ℓ growing proportionally to the rank with a sufficiently small coefficient of proportionality.

PROPOSITION 3.5. *Given $\delta > 0$, if $\delta < \frac{\ell}{n} < ec = \frac{e}{2}\log(\frac{1+\sqrt{5}}{2})$, the ℓ-entropy of a Cartan action of rank $n-1$ grows exponentially with n; the exponent is bounded below by the positive number $\frac{\ell}{n}\log c - \frac{\ell}{n}\log\frac{\ell}{n} + \frac{\ell}{n}$.*

The estimate for the Mahler measure (3.3) is not sufficient for estimating the Fried average entropy for Cartan actions. This estimate requires more sophisticated number-theoretic tools.

Consider the entropy norm in $\mathbb{R}^{n-1}$ associated with a Cartan action α given by commuting matrices $A_i, 1 \leq i \leq n-1$ with real eigenvalues $\lambda_j(A_i), 1 \leq j \leq n$. We define n Lyapunov hyperplanes on $\mathbb{R}^{n-1}$ by $\chi_j(\mathbf{t}) = 0$, $j = 1, \ldots, n$, where

$$\chi_j(\mathbf{t}) = \chi_j(t_1, \ldots t_{n-1}) = \sum_{i=1}^{n-1} t_i \log|\lambda_j(A_i)| \qquad (3.4)$$

for $1 \leq j \leq n$. In this case all hyperplanes are distinct. Since

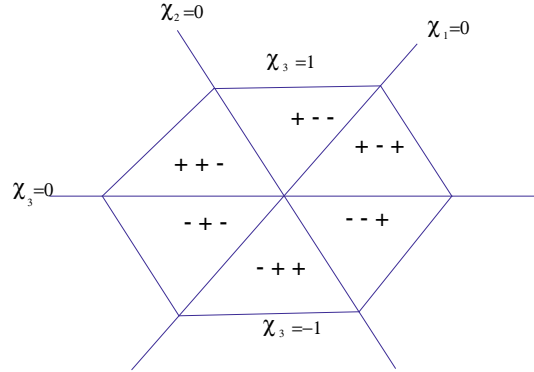$$\sum_{j=1}^{n} \log|\lambda_j(A_i)| = 0, \qquad (3.5)$$

Figure 1: Weyl chambers and the unit ball in the entropy function norm for $n = 3$

we have

$$\chi_1 + \chi_2 + \cdots \chi_n = 0, \tag{3.6}$$

hence inside each Weyl chamber the signs of χ_j's cannot be all the same, and the Lyapunov hyperplanes divide $\mathbb{R}^{n-1}$ into $2^n - 2$ Weyl chambers, in which the entropy function h_μ^α is linear. Hence the entropy function is a norm (for $n = 3$ Fig. 1).

In order to calculate the volume of $B(h_\mu^\alpha)$, we make change of variables $\varphi : \mathbb{R}^n \to \mathbb{R}^n$

$$y_j = \chi_j(t_1, \ldots t_{n-1}) - t_n, \quad 1 \leq j \leq n$$

which transforms the hyperplane $t_n = 0$ (corresponding to the original $\mathbb{R}^{n-1}$) into the hyperplane

$$\sum_{j=1}^{n} y_j = 0, \tag{3.7}$$

and the perpendicular line into the perpendicular line: $\varphi(0, \ldots, 0, 1) = (-1, \ldots, -1)$. By (2.2), the unit ball in the entropy norm $B(h_\mu^\alpha)$ is mapped into the intersection of the ball in $\mathbb{R}^n$ of ℓ^1-norm of radius 2 with the hyperplane (3.7) which we denote by H.

The Jacobian matrix of φ is

$$D = \begin{pmatrix} \log|\lambda_1(A_1)| & \log|\lambda_1(A_2)| & \cdots & \log|\lambda_1(A_{n-1})| & -1 \\ \log|\lambda_2(A_1)| & \log|\lambda_2(A_2)| & \cdots & \log|\lambda_2(A_{n-1})| & -1 \\ \cdots & \cdots & \cdots & \cdots & \\ \log|\lambda_n(A_1)| & \log|\lambda_n(A_2)| & \cdots & \log|\lambda_n(A_{n-1})| & -1 \end{pmatrix}.$$

By Proposition 3.2 we have $\lambda_i(A_j) = \varphi_i(\varepsilon_j)$, so if $\gamma(Z(A)) = U_K$, the matrix D with the last column removed is the transpose of the matrix that appears in the definition of the *regulator* R_K of the number field K [BS66, Chapter 2, Section 4].

Notice that the determinants of all $(n-1) \times (n-1)$ sub-matrices of D obtained by deleting the last column and any row are equal up to the sign. By (3.5) the rows of D with the last column removed are linearly dependent and sum to zero. This allows us to change the matrix D with the i-th row removed into the matrix D with the j-th row removed through a series of row operations (with $1 \le i, j \le n$). Since row operations affect only the sign of the determinant of a matrix, we conclude that the absolute value of the Jacobian $\frac{\partial(y_1,\ldots,y_n)}{\partial(t_1,\ldots,t_n)}$ is equal to nR, where R is the absolute value of the determinant of any such $(n-1) \times (n-1)$ sub-matrix, hence $dy_1 \wedge \cdots \wedge dy_n = nR\, dt_1 \wedge \cdots \wedge dt_n$. We write $dt_1 \wedge \cdots \wedge dt_n = ds \wedge dt_n$, where ds is the $(n-1)$-dimensional volume in the hyperplane $t_n = 0$, and similarly, $dy_1 \wedge \cdots \wedge y_n = ds' \wedge dt$, where ds' is the $(n-1)$-dimensional volume in the hyperplane H, and dt is in the perpendicular direction. Taking into account that the unit vector $(0,\ldots,0,1)$ is mapped to the vector $(-1,\ldots,-1)$ of length $\sqrt{n}$, i.e. that $dt = \sqrt{n}\, dt_n$, we conclude that $ds' = \sqrt{n}R\, ds$, and for the $(n-1)$-dimensional volumes we have

$$vol(H) = \int_H ds' = \sqrt{n}R \int_{B(h_\mu^\alpha)} ds = \sqrt{n}R\, vol(B(h_\mu^\alpha)).$$

Using the formula for the volume of the intersection of a unit ball in the ℓ^1-norm in $\mathbb{R}^n$ by a hyperplane passing through the origin, see [MP88, Prop. II.7], for the ball of radius 2 we have:

$$vol(H) = \sqrt{n}\binom{2n-2}{n-1}\frac{1}{(n-1)!}.$$

Therefore $vol(B(h_\mu^\alpha)) = \binom{2n-2}{n-1}\frac{1}{R(n-1)!}$, and

$$h^*(\alpha) = \frac{2^{n-1}}{(n-1)!\,vol(B(h_\mu^\alpha))} = \frac{R2^{n-1}}{\binom{2n-2}{n-1}}, \tag{3.8}$$

where

$$R = kR_K \tag{3.9}$$

and $k = [U_K : \gamma(\alpha)] \ge 1$. It thus follows that the lower bounds for the Fried average entropy will be found when $\gamma(\alpha) = U_K$.

Theorem 3.1. *The Fried average entropy $h^*(\alpha)$ of a Cartan action of a given rank $n-1 \ge 2$ is bounded away from zero by a positive function that grows exponentially with n,*

$$h^*(\alpha) > 0.000752 \exp(0.244n), \tag{3.10}$$

and, furthermore,

$$h^*(\alpha) \ge 0.089. \tag{3.11}$$

Proof. We use Zimmert's analytic lower bound for regulators [Zim81]. It states that for a totally real number field $[K : \mathbb{Q}] = n$, and any $s > 0$

$$R_K \geq a(s) \exp(b(s)n), \qquad (3.12)$$

where

$$a(s) = (1 + s)(1 + 2s) \exp\left(\frac{2}{s} + \frac{1}{1 + s}\right)$$

and

$$b(s) = \log\left(\frac{\Gamma(1 + s)}{2}\right) - (1 + s)\frac{\Gamma'}{\Gamma}\left(\frac{1 + s}{2}\right).$$

Due to (3.8), in order to obtain an exponential lower bound for the Fried average entropy we need $b(s) > \log 2$, and for $s = 0.35$ we obtain

$$R_K > 0.000376 \exp(0.9371n). \qquad (3.13)$$

Using the estimate for the middle binomial coefficient

$$\frac{4^{n-1}}{n} \leq \binom{2n - 2}{n - 1} \leq 4^{n-1} \qquad (3.14)$$

we obtain

$$\frac{1}{2^{n-1}} \leq \frac{2^{n-1}}{\binom{2n-2}{n-1}} \leq \frac{n}{2^{n-1}}.$$

Using the bound (3.13), we obtain

$$h^*(\alpha) = \frac{R_K 2^{n-1}}{\binom{2n-2}{n-1}} > \frac{0.000376 \exp(0.9371n)}{2^{n-1}} > 0.000752 \exp(0.244n), \quad (3.15)$$

thus proving the first assertion of Theorem 3.1, (3.10). The minimum of this lower bound (achieved for $n = 3$) is equal to $0.001565\ldots$ that is weaker than the claimed lower bound (3.11).

Inspection of the number fields data at http://www.lmfdb.org/ identifies the quartic totally real number field of discriminant 725 as the field that minimizes $h^*(\alpha)$ among the fields covered by the tables. For it $h^*(\alpha) = 0.330027\ldots$, and we conjecture that this gives the value $h_{\min}$ of $h^*(\alpha)$ for all Cartan actions α.

Although we are not able to prove this statement (see Conjecture 1), we can improve the lower bound to (3.11). Along the way we explain what number-theoretic tools are available for the actual proof that for any action α (i.e. for any totally real number field K)

$$h^*(\alpha) \geq h_{min},$$

and what the limitations presently are.

LEMMA 3.1. *The Fried average entropy $h^*(\alpha)$ of a Cartan action of rank $n-1$ for $3 \leq n \leq 7$ satisfies $h^*(\alpha) \geq h_{min} = 0.330027\ldots$.*

Proof. Suppose K is a totally real number field of degree n for which

$$h^*(\alpha) < h_{min}. \tag{3.16}$$

Then

$$\frac{0.000376 \exp(0.9371n)2^{n-1}}{\binom{2n-2}{n-1}} < 0.33002,$$

and by a direct numerical calculation, we find that $n \leq 16$.

Now we use Friedman's lower bound for the regulator of a totally real number field $[K : \mathbb{Q}] = n$ [Fri07],

$$R_K > 2g(1/D_K),$$

where

$$g(x) := \frac{1}{2^n 4\pi i} \int_{2-i\infty}^{2+i\infty} (\pi^n x)^{-s/2}(2s-1)\Gamma\left(\frac{s}{2}\right)^n ds.$$

It is known that $g(x)$ tends to $-\infty$ as $x \to 0+$, and that $g(x)$ is positive and vanishes exponentially fast for large x. If (3.16) holds, then $R_K < \frac{0.33002\binom{2n-2}{n-1}}{2^{n-1}}$ which implies that for some $c_1(n) < c_2(n)$ (computable numerically) $D_K < c_1(n)$ or $D_K > c_2(n)$. In order to use the upper bound $c_1(n)$ we need absolute upper bounds for D_K obtained by geometry of numbers [PZ89] that are $< c_2(n)$. For $n \leq 7$ such bounds are known, and the proof is completed by a finite check as follows. For $n = 7$ the Friedman's upper bound for D_K is smaller than the minimal discriminants of the totally real fields of this degree [Voi08]—so there are no totally real fields in degree 7. For degrees 3, 4, 5, 6 the Friedman's upper bounds for D_K are 115, 2,250, 40,400, and 710,000, respectively. The corresponding 19 totally real number fields found in http://www.lmfdb.org/ (2 of degree 3, 7 of degree 4, 4 of degree 5 and 6 of degree 6) are given in Tables 1, 2, 3 and 4. They show that the minimum of the Fried average entropy is achieved on the quartic field $K = \mathbb{Q}(\varepsilon)$ with discriminant 725 and the defining polynomial $x^4 - x^3 - 3x^2 + x + 1$, where ε is a fundamental unit. Using Pari-GP we find that for this field the index $[\mathcal{O}_K : \mathbb{Z}[\varepsilon]] = 1$, and by (3.2) we conclude that $\gamma(C(A)) = \mathcal{O}_K$, and hence $\gamma(Z(A)) = \mathcal{U}_K \cap \gamma(C(A)) = \mathcal{U}_K$.   □

Table 1 Totally real number fields of degree 3 with $D_K < 115$

D_K	f	R_K	$h^*(\alpha)$
49	$x^3 - x^2 - x + 1$	0.525454	0.350303
81	$x^3 - 3x - 1$	0.849287	0.566191

Table 2 Totally real number fields of degree 4 with $D_K < 2,250$

D_K	f	R_K	$h^*(\alpha)$
725	$x^4 - x^3 - 3x^2 + x + 1$	0.825068	0.330027
1,125	$x^4 - x^3 - 4x^2 + 4x + 1$	1.165455	0.466182
1,600	$x^4 - 6x^2 + 4$	1.542505	0.617002
1,957	$x^4 - 4x^2 - x + 1$	1.918363	0.767345
2,000	$x^4 - 5x^2 + 5$	1.852810	0.741124
2,048	$x^4 - 4x^2 + 2$	2.441795	0.976718
2,225	$x^4 - x^3 - 5x^2 + 2x + 4$	2.064511	0.825804

Table 3 Totally real number fields of degree 5 with $D_K < 40,400$

D_K	f	R_K	$h^*(\alpha)$
14,641	$x^5 - x^4 - 4x^3 + 3x^2 + 3x-$	1.635694	0.373873
24,217	$x^5 - 5x^3 - x^2 + 3x + 1$	2.399421	0.548439
36,497	$x^5 - x^4 - 3x^3 + 5x^2 + x - 1$	3.550657	0.811579
38,569	$x^5 - 5x^3 + 4x - 1$	3.155437	0.721243

Table 4 Totally real number fields of degree 6 with $D_K < 710,000$

D_K	f	R_K	$h^*(\alpha)$
300,125	$x^6 - x^5 - 7x^4 + 2x^3 + 7x^2 - 2x - 1$	3.277562	0.416198
371,293	$x^6 - x^5 - x^4 + 4x^3 + 6x^2 - 3x - 1$	3.774500	0.479302
434,581	$x^6 - 2x^5 - 4x^4 + 5x^3 + 4x^2 - 2x - 1$	4.187943	0.531802
453,789	$x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1$	4.399962	0.558725
592,661	$x^6 - x^5 - 5x^4 + 4x^3 + 5x^2 - 2x - 1$	4.525483	0.574665
703,493	$x^6 - 2x^5 - 5x^4 + 11x^3 + 2x^2 - 9x + 1$	5.233524	0.664574

Unfortunately, for $n > 7$ such bounds are not known, but for $8 \leq n \leq 16$ we can improve this minimum of the lower bound (3.15) by varying the parameter s in the Zimmert analytical formula (3.12).

Let $Z(n, s) = \frac{a(s)\exp(b(s))2^{n-1}}{\binom{2n-2}{n-1}}$ be the lower bound for $h^*(\alpha)$ obtained from the lower bound for the regulator (3.12). Using numerical calculation, we obtain

$$h^*(\alpha) \geq \min_n(\max_s Z(n, s)) = 0.089.$$

The functions $Z(n, s)$ for $n = 8, \ldots, 16, 17$ are shown in Fig. 2 (obtained by Mathematica). It is also evident from these plots that $n \leq 16$ is the best bound that can be obtained from Zimmert's analytic formula. □

CONJECTURE 1. The Cartan action α corresponding to the quartic totally real number field of discriminant 725 and the defining polynomial $x^4 - x^3 - 3x^2 + x + 1$ minimizes the Fried average entropy $h^*(\alpha)$ among all Cartan actions α. For that action $h^*(\alpha) = 0.330027... = h_{min}$.

Proposition 3.5 and Theorem 3.1 make it plausible that ℓ-entropies of Cartan actions grow exponentially with rank for all values of ℓ proportional to the rank.
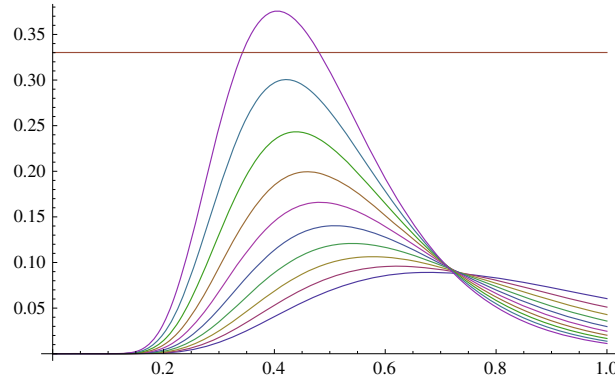
Figure 2: Plots of $Z(n, s)$ for $n = 8, \ldots, 16, 17$

Since for small coefficients of proportionality Proposition 3.5 applies we need only to consider coefficients larger than ec.

CONJECTURE 2. Let $\ell \geq \frac{e}{2} \log(\frac{1+\sqrt{5}}{2})n$. There exist positive constants a and b such that for any Cartan action α of rank $n - 1$

$$h^\ell(\alpha) > a \exp bn.$$

3.3 Rigidity of maximal rank actions.   The reason we pay such attention to Cartan actions is that they provide essentially the universal model for positive entropy maximal rank actions of $\mathbb{Z}^k, k \geq 2$. This is one of the principal results of [KR].

Theorem A.   *Let α be a C^r, $1 + \theta \leq r \leq \infty$ maximal rank positive entropy action on a smooth manifold M of dimension $n \geq 3$.*
     Then there exist:

- *disjoint measurable sets of equal measure $R_1, \ldots, R_m \subset M$ such that $R = \bigcup_{i=1}^m R_i$ has full measure and the action α cyclically interchanges those sets. Let $\Gamma \subset \mathbb{Z}^{n-1}$ be the stationary subgroup of any of the sets R_i (Γ is of course isomorphic to $\mathbb{Z}^{n-1}$);*
- *a Cartan action α_0 of Γ by affine transformations of either the torus $\mathbb{T}^n$ or the infratorus $\mathbb{T}^n/\{\pm \mathrm{Id}\}$ that we will call the algebraic model;*
- *measurable maps $h_i : R_i \to \mathbb{T}^n$ or $h_i : R_i \to \mathbb{T}^n/\{\pm \mathrm{Id}\}$, $i = 1, \ldots, m$;*

such that

1. *h_i is bijective almost everywhere and $(h_i)_*\mu = l$, the Lebesgue (Haar) measure on $\mathbb{T}^n$ (correspondingly $\mathbb{T}^n/\{\pm \mathrm{Id}\}$);*
2. *$\alpha_0 \circ h_i = h_i \circ \alpha$;*
3. *for almost every $x \in M$ and every $\mathbf{n} \in \mathbb{Z}^{n-1}$ the restriction of h_i to the stable manifold W_x^s of x with respect to $\alpha(\mathbf{n})$ is a $C^{r-\epsilon}$ diffeomorphism for any $\epsilon > 0$.*

4. h_i is $C^{r-\epsilon}$ in the sense of Whitney on a set whose complement to R_i has arbitrary small measure; they are saturated by local stable manifolds.

Notice that taking a finite factor does not change the entropy function and hence the value of Fried average entropy. Passing to a finite index subgroup multiplies the value of the Fried average entropy by the index of the subgroup, see Proposition 2.2(3). Since for a weakly mixing action $m = 1$, Theorem A, Proposition 3.1 and Theorem 3.1 imply the following rigidity property of the Fried average entropy.

COROLLARY 3.1. *The Fried average entropy of a weakly mixing maximal rank action of $\mathbb{Z}^k$, $k \geq 2$, is either equal to zero or is greater than 0.089; furthermore the lower bound grows with k exponentially, as in (3.10).*

From (3.8) and taking into account that for a general ergodic action the Fried average entropy may only be a rational multiple of that for the weakly mixing ergodic components of its restriction to a certain finite index subgroup, we also have

COROLLARY 3.2. *The Fried average entropy of a maximal rank action of $\mathbb{Z}^k$, $k \geq 2$, may take only countably many values.*

4 Connection Between the Fried Average Entropy and Slow Entropy

4.1 Refining the slow entropy type invariants. Slow entropies for actions of higher rank abelian groups have been defined and discussed in [KT97]. As we mentioned in the introduction, the motivation for using slow entropies for smooth actions of higher rank abelian groups is as follows: the number of different codes with respect to a nice partition for any smooth action grows no faster than exponentially against the size of the ball in a group, i.e. the logarithm of this number grows at most linearly. Hence in order to produce a non-vanishing invariant logarithms of those numbers need to be normalized not by the number of elements in the ball but by its kth root where, in the case of abelian groups, k is the rank of the acting group. This corresponds to the slow entropy with the scale function $s^{1/k}$.

One needs to elaborate the constructions from [KT97, Section 1.3]. First, we will restrict ourselves to the case of $\mathbb{Z}^k$ actions that are the only one of interest in the context of this paper. In this context the Følner sequence that appears in the discussion will be a sequence of balls in a metric in $\mathbb{Z}^k$ induced by a norm in the ambient space $\mathbb{R}^k$. Neither of two specific suggestions from [KT97, Section 1.3] produce numerical invariants that distinguish maximal rank full entropy actions. Other choices are alluded to in the sentence: "The reader will find without much difficulty other convenient characteristics of the asymptotic growth". We will describe one such choice shortly. This will produce a numerical invariant that depends on a choice of a norm in $\mathbb{R}^k$ and in particular, depends on a choice of generators for a $\mathbb{Z}^k$ action. To address this, we will minimize the value of this norm-dependent slow entropy

over all choices of norm with the unit ball of volume one. We will show that in the case we are interested, namely the case (**P**) of the alternative in section 2 for affine actions on the torus and Lebesgue measure the minimum is always positive and is achieved for a certain polyhedral norm. By extending this definition to $\mathbb{R}^k$ actions and connecting it with the quantities that appear in the Ledrappier–Young formula for entropy one can show that the resulting notion satisfies conditions (i)–(iv) from the introduction and that the statement about polyhedral minimizers also holds in general. Detailed treatment of the general case will appear in a forthcoming paper by Changguang Dong.

In the present paper we will restrict specific calculations to the case of Cartan actions. We will show that in each dimension the values of the Fried average entropy and the slow entropy determine each other, that via Theorem A will imply corresponding results for arbitrary maximal rank actions.

For a moment we return to a general situation. Let α be an action of a discrete group Γ by transformation of a space X preserving measure μ. All slow entropy constructions begin with defining the essential number of substantially different orbit segments or blocks. A block is defined by an initial condition $x \in X$ and a finite set $F \subset \Gamma$ it is simply $\{\alpha(\gamma)x, \gamma \in F\}$. The word "essential" means that sets of fixed small measure can be ignored. "Substantially different" is defined by using a family of metrics or semi-metrics d_F in the spaces of F-blocks and considering blocks with d_F distances greater that a fixed small number as essentially different. There are two standard ways to define d_F. One is based on an idea of coding. One fixes a finite partition ξ of the space X into measurable sets $c_1, \ldots, c_N$ and defines d_F as the pull-back of the Hamming metric in the space of codes, see [KT97, Section 1.1] for details. In the other method one assumes that α is an action by homeomorphisms of a compact metrizable space, starts with a metric d in X and defines $d_F = \max_{\gamma \in F} d \circ \alpha(\gamma)$. Then one defines $S_{d_F}(\alpha, \epsilon, \delta)$ as the minimal number of d_F balls whose union has measure $\geq 1 - \delta$. To make notations more suggestive this quantity in the first of the above cases is denoted by $S_\xi^H(\alpha, F, \epsilon, \delta)$ and in the second by $S_d(\alpha, F, \epsilon, \delta)$.

Now we go back to the abelian case. Let α be an action of $\mathbb{Z}^k$. Given a norm p on $\mathbb{R}^k$, let F_s^p be the intersection of the lattice $\mathbb{Z}^k$ with the ball centered in 0 and radius s. We define the slow entropy of α with respect to the norm p and the partition ξ as

$$sh(p, \alpha, \xi) = \lim_{\epsilon, \delta \to 0} \limsup_{s \to \infty} \frac{1}{s} \log S_\xi^H(\alpha, F_s^p, \epsilon, \delta). \tag{4.1}$$

Then we define

$$sh(p, \alpha) = \sup_\xi sh(p, \alpha, \xi). \tag{4.2}$$

[KT97, Proposition 1] states that slow entropy can be defined using any generating sequence of partition, e.g. in the case of an action by homeomorphisms any sequence of partitions such that the maximal diameter on an element converges to zero. [KT97, Proposition 2] then implies that

$$sh(p, \alpha) = \lim_{\epsilon, \delta \to 0} \limsup_{s \to \infty} \frac{1}{s} \log S_d(\alpha, F_s^p, \epsilon, \delta). \tag{4.3}$$

Notice that since (4.2) is a measure-theoretic isomorphism invariant the limit in (4.3) does not depend on the choice of metric d.

Finally we define the slow entropy of α as

$$sh(\alpha) = \inf_{p: vol(p)=1} sh(p, \alpha)$$

where $vol(p)$ is the volume of the unit ball in the norm p (we are assuming $\mathbb{Z}^k \subset \mathbb{R}^k$ has co-volume 1). If α is understood, we will omit reference to it like $sh(p) = sh(p, \alpha)$.

REMARK 2. Taking the logarithm in (4.1) is essential for obtaining a numerical invariant that takes finite positive values in many cases. This is not true for two suggestions for asymptotic invariants in [KT97, Section 1.3]. The first of those may be too subtle and the second is too crude. A good analogy between the latter and our definition is between the Hausdorff dimension of a set and Hausdorff measure corresponding to a fixed value of Hausdorff dimension.

4.2 Slow entropy and Lyapunov exponents. Let α be a $\mathbb{Z}^k$ action by affine maps of $\mathbb{T}^n$. Let $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_r$ be the splitting into coarse Lyapunov subspaces of this linear action and let us fix once and for all the max norm $|\cdot|$ on $\mathbb{R}^n$ with respect to the splitting $E_1 \oplus \cdots \oplus E_r$ i.e. if $v = v_1 + \cdots + v_r$ with respect to the splitting $E_1 \oplus \cdots \oplus E_r$ then $|v| = \max_i |v_i|$.

Let p be a norm on $\mathbb{R}^k$ such that the unit ball has volume one. We want to compute $sh(p, \alpha)$. We will use (4.3) for the metric on the torus defined by the above defined norm $|\cdot|$. Thus the metric $d_{F_s^p}$ is defined by the norm

$$|v|_s^p = \max_{t: p(t) \leq s} |\alpha(t)v|.$$

Let B_s^p be the unit ball in the norm $|\cdot|_s^p$.

LEMMA 4.1.

$$sh(p, \alpha) = \lim_{s \to \infty} -\frac{1}{s} \log vol(B_s^p). \tag{4.4}$$

Proof. Due to translation invariance of the metric $d_{F_s^p}$ all balls of a given radius have the same volume that for a small enough ϵ and all s is equal to $\epsilon^n vol(B_s^p)$. Hence

$$S_d(\alpha, F_s^p, \epsilon, \delta) \geq \frac{1 - \delta}{\epsilon^n vol(B_s^p)}$$

that gives a lower bound independent of ϵ and δ.

To get an upper bound notice that the parallelepiped ϵB_s^p tiles $\mathbb{R}^n$ and hence the tiles provide a disjoint cover of a fundamental domain for $\mathbb{T}^n$ except for a neighborhood of the boundary of the size $\epsilon \, diam(B_s^p)$. In particular, for a small enough ϵ the tiles cover an area of volume greater then one half. $\qquad \square$

Now, $v \in B_s^p$ if and only if $|v|_s^p = \max_{t:p(t) \leq s} |\alpha(t)v| \leq 1$ and $vol(B_s^p)$ is up to a constant the product of the corresponding $vol(B_s^p \cap E_i)$. Let $J_i(t)$ be the Jacobian of $\alpha(t)$ along E_i and let $\hat{\chi}_i(t) = \log |J_i(t)|$. $\hat{\chi}_i$ is proportional to any Lyapunov exponent appearing in the coarse Lyapunov direction E_i.

Let

$$M_i^p(s) = \max_{t:p(t) \leq s} \hat{\chi}_i(t).$$

Then we have that (up to a constant)

$$vol(B_s^p) = \prod_i \exp(-M_i^p(s)).$$

Observe that since $\hat{\chi}_i$ and p are $1-$homogeneous we get that $M_i^p(s) = sM_i^p(1)$ hence

$$-\frac{1}{s} \log vol(B_s^p) = \sum_i M_i^p(1) = \sum_i \max_{t:p(t) \leq 1} \hat{\chi}_i(t).$$

Thus for any action α by affine maps of a torus the slow entropy for Lebesgue measure with respect to the norm p is equal to

$$sh(p, \alpha) = \sum_i \max_{t:p(t) \leq 1} \hat{\chi}_i(t) = \sum_i \max_{t:p(t) \leq 1} \chi_i(t) = \sum_i p^*(\hat{\chi}_i) \qquad (4.5)$$

where p^* is the dual norm to p, i.e. $p^*(u) = \max_{p(t)=1} |u(t)|$.

Notice that since by the Pesin entropy formula $h(\alpha(t)) = \frac{1}{2} \sum_i |\chi_i(t)|$ [see (2.2)] and hence

$$sh(p, \alpha) \geq 2 \max_{t:p(t) \leq 1} h(\alpha(t)). \qquad (4.6)$$

Now we want to minimize this over all norms p with unit ball of volume 1.

First, notice that there is dichotomy that already appeared in section 2. Namely either all elements of the suspension, save for identity, have positive entropy, or a nontrivial element of the suspension has zero entropy. Since we are dealing with actions preserving Lebesgue measure, by the Pesin entropy formula the second alternative is equivalent to non-triviality of the intersection of all Lyapunov hyperplanes, i.e. existence of a common annihilator for all Lyapunov exponents. First we show, that, similarly to the case of Fried average entropy, this dichotomy is equivalent to the zero-positive dichotomy for the slow entropy.

PROPOSITION 4.1. *For an affine action α on a torus $sh(\alpha) = 0$ if and only if intersection of all Lyapunov hyperplanes contains a non-zero vector, or, equivalently, the entropy function is not a norm, or, equivalently, Fried entropy vanishes.*

Proof. First, let's assume that the entropy function is a norm. This implies that the entropy reaches a positive minimum m on the unit sphere. Consider the unit ball of volume one with respect to a norm p. It is a convex centrally symmetric body and hence contains a vector t of length greater than a certain absolute constant C. Hence by (4.6) $sh(p, \alpha) \geq 2mC$.

Now assume that for some $t \neq 0$, $\chi_i(t) = 0$ for all i. Fix an $\epsilon > 0$ and consider a norm p_ϵ whose unit ball is the product of a segment of the line generated by t and the ball of radius ϵ in the hyperplane perpendicular to t. Notice that all Lyapunov exponents of α are bounded in absolute value on the unit ball of the norm p_ϵ by $\epsilon \cdot const.$ □

Thus we only need to consider the case of positive slow entropy. It appears exactly when the Lyapunov exponents (and hence the vectors $\hat{\chi}_i$) generate the dual space to $\mathbb{R}^n$.

4.3 Convex bodies and minimizers for affine actions. Let $\xi_1, \ldots, \xi_n \in \mathbb{R}^k$ be a set of spanning vectors and let C be the symmetric convex hull of the $\xi_i's$, i.e. C is the convex hull of the set $\{\xi_1, \ldots, \xi_n, -\xi_1, \ldots, -\xi_n\}$. Let p_C be the Minkowski functional associated to C. Remember that given a norm p, we denote its dual norm by p^*, and that $(p^*)^* = p$. Given a convex symmetric body, C let

$$C^\circ = \{x : |x \cdot \xi| \leq 1 \text{ for any } \xi \in C\}.$$

Observe that $p_C^* = p_{C^\circ}$.

LEMMA 4.2.

$$p_C^*(v) = \max_i |\xi_i \cdot v|.$$

Proof. We claim that

$$C^\circ = \bigcap_i \{x : |x \cdot \xi_i| \leq 1\}.$$

Indeed, let $A = \bigcap_i \{x : |x \cdot \xi_i| \leq 1\}$. By definition of C° we have that $C^\circ \subset A$. We shall use the following simple property.

If $C^\circ \subset A$ then $A^\circ \subset C$ and $C^\circ = A$ if and only if $A^\circ = C$.

We have then that $A^\circ \subset C$, let us prove that $C \subset A^\circ$. Let $\xi \in C$. Then

$$\xi = \sum_i t_i^+ \xi_i - \sum_i t_i^- \xi_i = \sum_i (t_i^+ - t_i^-) \xi_i,$$

where $\sum_i t_i^+ + \sum_i t_i^- = 1$, $t_i^\pm \geq 0$.

Take $x \in A$, then $|x \cdot \xi_i| \leq 1$ for any i and hence

$$|x \cdot \xi| = \left| x \cdot \sum_i (t_i^+ - t_i^-) \xi_i \right| = \left| \sum_i x \cdot (t_i^+ - t_i^-) \xi_i \right|$$

$$\leq \sum_i |t_i^+ - t_i^-| |x \cdot \xi_i| \leq \sum_i t_i^+ + t_i^- = 1.$$

So, we get that $\xi \in A^\circ$ and hence

$$C^\circ = \bigcap_i \{x : |x \cdot \xi_i| \leq 1\}.$$

Now, for this C° the Minkowski functional is

$$p_{C^\circ}(v) = p_C^*(v) = \max_i |\xi_i \cdot v|. \qquad \qquad \square$$

Given a norm, we denote by $vol(p)$ the volume of the unit ball in the norm p.

Instead of minimizing the quantity

$$\sum_i p^*(\xi_i)$$

over the norms p with $vol(p) = 1$ we shall minimize the following more homogeneous quantity. Given a family of functionals $\Xi = (\xi_1, \dots, \xi_n)$ and a norm p in $\mathbb{R}^k$ we define

$$SH_\Xi(p) = \frac{\sum_i p^*(\xi_i)}{vol(p)^{\frac{1}{k}}}.$$

This last quantity is invariant by $c \mapsto cp$, hence this is the same.

PROPOSITION 4.2. *Given a spanning set of functionals $\Xi = (\xi_1, \dots, \xi_n)$ the minimum of $SH_\Xi(p)$ is attained at a norm of the form $q(x) = \max_i c_i |\xi_i \cdot x|$ for some values of $c_i \geq 0$.*

Proof. Let p be a norm. Let us take C the symmetric convex hull of

$$\left\{ \frac{\xi_i}{p^*(\xi_i)} \right\}_i.$$

Let q^* be the Minkowski functional of C and q its dual norm. Observe that $C \subset B^{p^*}(1)$ (the unit ball of p^*). By definition we have that

$$q^*(\xi_i) \leq p^*(\xi_i)$$

for every i. Also we have that the unit ball of q is C°. And hence $B^p(1) = (B^{p^*}(1))^\circ \subset C^\circ = B^q(1)$ and hence

$$vol(p) \leq vol(q).$$

Hence we get that

$$SH_\Xi(q) \leq SH_\Xi(p).$$

Since C is a convex polyhedron the norm q has the form

$$\max_i c_i |\xi_i \cdot x|. \qquad \qquad (4.7)$$

Now by homogeneity we may without loss of generality assume that $\sum_i c_i = 1$ and consider the function SH_Ξ on the unit simplex. Notice that SH_Ξ is defined if and only if (4.7) is a norm. This is true if all coefficients c_1, $i = 1, \dots, n$ are positive, i.e in the interior of the simplex, as well as for those points on the boundary where the vectors ξ_i with non-zero coefficients form a spanning set. The function SH_Ξ is continuous at all points of the simplex where it is defined and goes to infinity elsewhere. Hence it reaches a positive minimum. $\qquad \square$

Next we will compute how SH_Ξ behaves under linear change of variable.

LEMMA 4.3. *Given an invertible linear map L on $\mathbb{R}^k$, a norm q on $\mathbb{R}^k$ and linear functionals $\Xi = (\xi_1, \ldots, \xi_n)$ in $\mathbb{R}^k$, let $\eta_i = (L^t)^{-1}\xi_i$, $E = (\eta_1, \ldots, \eta_n)$ and $L_*q(x) = \hat{q}(x) = q(Lx)$. Then*

$$SH_\Xi(L_*q) = SH_\Xi(\hat{q}) = SH_E(q)|\det L|^{1/k}.$$

Proof. Let $\hat{q}(x) = q(Lx)$. Then $\hat{q}^*(\xi) = q^*((L^t)^{-1}\xi)$ and $L(B^{\hat{q}}(1)) = B^q(1)$. Hence

$$\det(L)vol(\hat{q}) = vol(q).$$ $\square$

4.4 Slow entropy for Cartan actions.

Theorem 4.1. *Let α be a Cartan action on the torus $\mathbb{T}^n$ then*

$$sh(\alpha) = C(n)R^{\frac{1}{n-1}} \tag{4.8}$$

where $R = kR_K$ is as in (3.9), i.e. R_K is the regulator and $k = [U_K : \gamma(\alpha)]$ and

$$\frac{n-1}{2} \le C(n) \le n-1. \tag{4.9}$$

Proof. Let χ_i, $i = 1, \ldots, n$ be linear functionals in $\mathbb{R}^{n-1}$ corresponding to the Lyapunov exponents of a Cartan action α. Hence $\sum_{i=1}^n \chi_i = 0$ and $\chi_1, \ldots \chi_{n-1}$ are linearly independent.

Let $\hat{q}(x) = \max_i c_i|\chi_i \cdot x|$ for some $c_i \ge 0$. Notice that in order for $\hat{q}$ to be a norm at most one coefficient c_i may vanish. Thus, without loss of generality, we may assume that $c_i > 0$ for $i < n$. Now let L be a linear transformation such that $(L^t)^{-1}(c_i\chi_i) = e_i$, $i = 1, \ldots, n-1$, where e_i is the canonical basis in $\mathbb{R}^{n-1}$. Then we have by Lemma 4.3 that

$$SH_{\chi_1,\ldots,\chi_n}(\hat{q}) = SH_{\frac{e_1}{c_1},\ldots,\frac{e_{n-1}}{c_{n-1}},\eta}(q)|\det L|^{1/k}$$

where $\eta = (L^t)^{-1}\chi_n = -\sum_{i=1}^{n-1}\frac{e_i}{c_i}$ and

$$q(x) = \hat{q}(L^{-1}x) = \max_i c_i|\chi_i \cdot L^{-1}x| = \max_i c_i|(L^t)^{-1}\chi_i \cdot x|$$

$$= \max\left(\max_{1\le i\le n-1} c_i|\frac{e_i}{c_i} \cdot x|, c_n|(L^t)^{-1}\chi_n \cdot x|\right)$$

$$= \max\left(\max_{1\le i\le n-1} |x_i|, c_n|\eta \cdot x|\right).$$

Finally,

$$SH_{\chi_1,\ldots,\chi_n}(\hat{q}) = SH_{\frac{e_1}{c_1},\ldots,\frac{e_{n-1}}{c_{n-1}},\eta}(q)|\det L|^{\frac{1}{n-1}}$$

$$= \frac{\sum_{i=1}^{n-1}\frac{q^*(e_i)}{c_i} + q^*(\eta)}{vol(q)^{\frac{1}{n-1}}}\left(\prod_{i=1}^{n-1} c_i\right)^{\frac{1}{n-1}} R^{\frac{1}{n-1}} \tag{4.10}$$

where $R = |\det(\chi_i)_{1\leq i\leq n-1}|$ as in (3.9) is the product of the regulator R_K and $k = [U_K : \gamma(\alpha)]$.

At this point we already see that values of $c_1, \ldots, c_n$ for which the minimum of $SH_{\chi_1,\ldots,\chi_n}(\hat{q})$ is reached do not depend on $\chi_1, \ldots, \chi_n$. Hence (4.10) implies (4.8).

We can assume without loss of generality that $c_n \leq c_i$ for every i, and let $\hat{\eta} = c_n \eta$. Then we have that $\hat{\eta} = \left(-\frac{c_n}{c_1}, \ldots, -\frac{c_n}{c_{n-1}}\right)$ and all its coordinates are of absolute value smaller than or equal to 1. Thus

$$q(x) = \max\left(\max_{1\leq i\leq n-1} |x_i|\right), |\hat{\eta}\cdot x|$$

and we need to find upper and lower bounds for the minimum of

$$\frac{\sum_{i=1}^{n-1} \frac{q^*(e_i)}{c_i} + \frac{q^*(\hat{\eta})}{c_n}}{vol(q)^{\frac{1}{n-1}}} \left(\prod_{i=1}^{n-1} c_i\right)^{\frac{1}{n-1}}$$

over $(c_1, \ldots, c_n)$; more specifically, in order to prove (4.9) we need to show that the minimum in question is bound between $\frac{n-1}{2}$ and $n-1$.

LEMMA 4.4. $q^*(e_i) = 1$ for $i = 1, \ldots, n-1$.

Proof.

$$q^*(e_i) = \max_{q(x)\leq 1} |x_i|.$$

By the definition of $q(x)$ we get that if $q(x) \leq 1$ then $|x_i| \leq 1$; moreover, taking $x = e_i$ we get that $|x_i| = 1$, $x_j = 0$ for $j \neq i$ and $|\hat{\eta}\cdot x| = \frac{c_n}{c_i} \leq 1$. Hence $q(e_i) = 1$ and $q^*(e_i) = 1$.   □

LEMMA 4.5. $q^*(\hat{\eta}) = \min\{1, \sum_{i=1}^{n-1} \frac{c_n}{c_i}\}$.

Proof. Clearly $q^*(\hat{\eta}) \leq 1$. Let $\bar{1} = (1, 1, \ldots, 1)$ be the vector with all coordinates equal to 1, i.e. $\bar{1} = \sum e_i$. Then $|\hat{\eta}\cdot\bar{1}| = \sum_{i=1}^{n-1} \frac{c_n}{c_i}$, hence $q(\bar{1}) = \max\{1, |\hat{\eta}\cdot\bar{1}|\}$. So, if $q(\bar{1}) = 1$ then $1 \geq |\eta\cdot\bar{1}|$ and hence we get the claim. If $q(\bar{1}) = |\eta\cdot\bar{1}|$ and then $|\eta\cdot\bar{1}| \geq 1$.   □

So there are two cases:

Case I: $q^*(\hat{\eta}) = \sum_{i=1}^{n-1} \frac{c_n}{c_i}$.

Here $|\hat{\eta}\cdot\bar{1}| \leq 1$. Hence, if $|x_i| \leq 1$ for any i then $|\hat{\eta}\cdot x| \leq |\hat{\eta}\cdot\bar{1}| \leq 1$ and consequently q is the l^∞ norm,

$$q(x) = \max_i |x_i|.$$

In particular, $vol(q) = 2^{n-1}$, hence

$$\frac{\sum_{i=1}^{n-1} \frac{q^*(e_i)}{c_i} + \frac{q^*(\hat{\eta})}{c_n}}{vol(q)^{1/k}} \left(\prod_{i=1}^{n-1} c_i\right)^{\frac{1}{n-1}} = \sum_{i=1}^{n-1} \frac{1}{c_i} \left(\prod_{i=1}^{n-1} c_i\right)^{\frac{1}{n-1}}.$$

The right hand side is always larger than or equal to $(n-1)$ and it minimizes precisely when $c_i = c$ for every $i = 1, \ldots, n-1$ and the value of c_n is arbitrary as long as $c_n \leq \frac{c}{n-1}$. This gives the above estimate in (4.9) and hence upper bound for $sh(\alpha)$.

Case II: $q^*(\hat{\eta}) = 1$.

Here $|\hat{\eta} \cdot \bar{1}| \geq 1$. So the unit ball for q is the unit cube intersected with the slice $|\hat{\eta} \cdot x| \leq 1$ and

$$
\frac{\sum_{i=1}^{n-1} \frac{q^*(e_i)}{c_i} + \frac{q^*(\hat{\eta})}{c_n}}{vol(q)^{\frac{1}{n-1}}} \left(\prod_{i=1}^{n-1} c_i \right)^{\frac{1}{n-1}} = \frac{\sum_{i=1}^{n} \frac{1}{c_i}}{vol(q)^{\frac{1}{n-1}}} \left(\prod_{i=1}^{n-1} c_i \right)^{\frac{1}{n-1}}.
$$

Now we have that

$$
\frac{\sum_{i=1}^{n} \frac{1}{c_i}}{vol(q)^{\frac{1}{n-1}}} \left(\prod_{i=1}^{n-1} c_i \right)^{\frac{1}{n-1}} \geq \frac{\sum_{i=1}^{n-1} \frac{1}{c_i}}{vol(q)^{\frac{1}{n-1}}} \left(\prod_{i=1}^{n-1} c_i \right)^{\frac{1}{n-1}} \geq \frac{n-1}{vol(q)^{\frac{1}{n-1}}} \geq \frac{n-1}{2}.
$$

The last inequality follows since the unit ball of q is inside the unit cube. This gives the below estimate in (4.9) and hence the lower bound for $sh(\alpha)$. $\qquad \square$

In light of good upper and and lower bounds (4.9) for the value of slow entropy it looks like finding the actual minimizer and hence exact value of the constant $C(n)$ in Theorem 4.1 will not much improve our understanding of the slow entropy and its connection with the Fried average entropy. Still it would be nice to know the minimizer and exact value of the $C(n)$ for aesthetic reasons. Evidence from the low-dimensional cases as well as symmetry arguments support the assertion that the minimizer appears for $c_1 = \cdots = c_n$.

CONJECTURE 3. The minimum in (4.10) is attained at the norm $q(x) = \max_i |\chi_i \cdot x|$.

Similarly to the derivation of Corollary 3.1, we obtain from Theorem A, formula (3.8), Theorem 3.1 and Theorem 4.1:

COROLLARY 4.1. *The slow $sh(\alpha)$ entropy of a weakly mixing maximal rank action α of $\mathbb{Z}^k$, $k \geq 2$ is expressed through its Fried average entropy $h^*(\alpha)$ as follows*

$$
sh(\alpha) = c(k) \binom{2k}{k}^{\frac{1}{k}} (h^*(\alpha))^{\frac{1}{k}},
$$

where $k/4 \leq c(k) \leq k/2$. In particular, the slow entropy is either equal to zero or uniformly bounded away from zero; the lower bound grows linearly with the rank.

A slight variation appears in the case of ergodic but not weakly mixing actions because of the normalization that is fully determined by the periodic part of the action but may not be uniform.

Acknowledgments

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