Measure rigidity for actions of higher–rank abelian groups

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References

[F-1967] H. Furstenberg., Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory*, **1**(1967), 1–49 .

[Ru90] D. Rudolph, $\times 2$ and $\times 3$ invariant measures and entropy *Ergodic Theory Dynam. Systems* **10** (1990), no. 2, 395–406.

[KS96] A. Katok and R. Spatzier, Invariant measures for higher-rank hyperbolic abelian actions, *Ergodic Theory Dynam. Systems* **16** (1996), 751–778.

[KaK99] B. Kalinin and A. Katok, Invariant measures for actions of higher rank abelian groups, in *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, Amer. Math. Soc., Providence, RI, 2001, pp. 593–637.

[KaS02] B. Kalinin and R. Spatzier, Measurable rigidity for higher rank abelian actions, to appear in *Ergodic Theory Dynam. Systems*.

[EK02] M. Einsiedler and A. Katok, Invariant measures on G/Γ for split simple Lie-groups G, *Comm. Pure Appl. Math.*, **56**, (2003), 1184–1221.

[EKL03] M. Einsiedler, A. Katok, and E. Lindenstrauss, Invariant measures and the set of exceptions to Littlewood's conjecture, preprint, 2003. $http://www.math.psu.edu/katok\_a/papers.html$

[EK04] M. Einsiedler and A. Katok, Rigidity of measures – the high entropy case and non-commuting foliations, preprint, 2004. $http://www.math.psu.edu/katok\_a/papers.html$

ACTIONS of $\mathbb{Z}_+^k, \mathbb{Z}^k, \mathbb{R}^k,\ k \geq 2$.

BASIC EXAMPLES

(1). $\times 2, \times 3$ (Furstenberg,[F-1967]):

$$E_2 : S^1 \to S^1 \quad x \mapsto 2x, \quad (\mod 1)$$
$$E_3 : S^1 \to S^1 \quad x \mapsto 3x, \quad (\mod 1).$$

(2). *Commuting toral automorphisms*:
$A, B \in SL(3, \mathbb{Z}),\ \ AB = BA,\ A^k = B^l \to k = l = 0,\ A, B$ hyperbolic. The $\mathbb{Z}^2$ action generated by automorphisms of the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$:

$$F_A : x \mapsto Ax, \quad (\mod 1)$$
$$F_B : x \mapsto Bx, \quad (\mod 1).$$

(3). *Weyl chamber flow (WCF)*: $M = SL(n, \mathbb{R})/\Gamma,\ n \geq 3, \Gamma$ a lattice in $SL(n, \mathbb{R}),\ D$ positive diagonals isomorphic to $\mathbb{R}^{n-1}$.
WCF:

The action of D on M by left translations.

GENERAL ALGEBRAIC ANOSOV ACTIONS

- An action of a group G on a manifold is *Anosov* if some element $g \in G$ acts normally hyperbolically with respect to the orbit foliation.

- It is *partially hyperbolic* if there is an invariant foliations $\mathcal{F}$ and an element which acts normally hyperbolically with respect to $\mathcal{F}$

Let H be a connected Lie group with $\Lambda \subset H$ a cocompact lattice. Define Aff (H) as the set of diffeomorphisms of H which map right invariant vectorfields on H to right invariant vectorfields. Let Aff (H/Λ) be the diffeomorphisms of H/Λ which lift to elements of Aff (H).

- An action ρ of a discrete group G on H/Λ is *affine algebraic* if $\rho(g)$ is given by a homomorphism $G \to \text{Aff}\,(H/\Lambda)$.

The linear part: Let $\mathfrak{h}$ be the Lie algebra of H. Identifying $\mathfrak{h}$ with the right invariant vectorfields on H, any affine algebraic action determines a homomorphism $\sigma : G \to \mathrm{Aut}\ \mathfrak{h}$. Call σ the *linear part* of this action. We will also allow quotient actions of these on finite quotients of H/Λ, e.g. on infranilmanifolds.

According to the properties of the linear part the algebraic actions can be classified into:

- *Elliptic* (semisimple eigenvalues of modulus one),
- *Parabolic* (eigenvalues of modulus one with some Jordan blocks),
- *Hyperbolic* (Anosov); if G is discrete H has to be nilpotent,
- *Partially hyperbolic* (otherwise).

Automorphisms of the tori and solenoids; generalization of basic examples NN 1,2: If H is abelian then the quotient H/Λ ia a torus and $\mathbb{Z}^k$ acts by automorphisms.

An automorphism of a torus acts ergodically with respect to Haar measure iff there are no roots of unity among its eigenvalues. Any such automorphism is Anosov or partially hyperbolic.

The *genuine higher rank assumption* for an action of $\mathbb{Z}^k$ on a torus:

$\mathbb{Z}^k$ contains a $\mathbb{Z}^2$ such that every non-trivial element of $\mathbb{Z}^2$ acts ergodically with respect to Haar measure.

The *natural extension* of an action by toral *endomorphisms* (as in the basic example 1) is an acion of automorphisms by a *solenoid*.

Homogeneous actions:

Let $\mathbb{R}^k \subset H$ where H is a connected Lie group. Let $\mathbb{R}^k$ act on a quotient H/Λ by left translations where Λ is a lattice in H. Suppose C is a compact subgroup of H which commutes with $\mathbb{R}^k$. Then the $\mathbb{R}^k$-action on H/Λ descends to an action on $C \setminus H/\Lambda$. The general algebraic $\mathbb{R}^k$-action ρ is a finite factor of such an action. Let $\mathfrak{c}$ be the Lie algebra of C. The *linear part* of ρ is the representation of $\mathbb{R}^k$ on $\mathfrak{c} \setminus \mathfrak{h}$ induced by the *adjoint representation* of $\mathbb{R}^k$ on the Lie algebra $\mathfrak{h}$ of H.

Suspension

The suspension construction allows to associate with an algebraic $\mathbb{Z}^k$ action α an algebraic $\mathbb{R}^k$ action called *the suspension action* whose dynamical properties are very closely related to those of the action α.

- The suspension of an Anosov action is Anosov.
- The suspension of an action by automorphisms of a torus is a homogeneous action on a certain solvable Lie group.

Symmetric space examples (generalization of the basic example N3):

- G a semisimple connected real Lie group of the noncompact type and of $\mathbb{R}$-rank at least 2;
- A the connected component of a split Cartan subgroup of G.
- Γ is an irreducible torsion-free cocompact lattice in G.

The centralizer $Z(A)$ of A splits as a product $Z(A) = M\,A$ with M compact. Since A commutes with M, the action of A by left translations on G/Γ descends to an A-action on $N =: M \setminus G/\Gamma$. This is a general *Weyl chamber flow.*

Any Weyl Chamber flow is Anosov.

If G is $\mathbb{R}$-*split* (e.g. $G = SL(n, \mathbb{R})$) then $M = \{1\}$.

LYAPUNOV EXPONENTS

Let μ be an α-invariant measure; by the natural extension and suspension we can always reduce the situation to the case of $\mathbb{R}^k$.

- *Lyapunov exponents:* $\chi_i \in (\mathbb{R}^k)^*$;
- *Lyapunov hyperplanes:* $\operatorname{Ker} \chi_i$;
- *Weyl chambers.* Connected components of $\mathbb{R}^k \setminus \bigcup_i \operatorname{Ker} \chi_i$;

LYAPUNOV FOLIATIONS

• For a Lyapunov exponent χ a (positive) *Lyapunov half–space* is $\chi^{-1}(0,\infty)$. Lyapunov exponents have the same Lyapunov half–spaces iff they are positively proportional.

• For an algebraic action each Lyapunov exponent defines a smooth (right G–invariant) *Lyapunov distribution.*

• The sum of the distributions corresponding to all positively proportional exponents is called a *coarse Lyapunov distribution* (identified with a Lyapunov half–space).

• A coarse Lyapunov distribution is integrable to a smooth (right–invariant) *Lyapunov foliation.* It is a minimal nontrivial intersection of unstable foliations for certain elements of the action.

These foliations are basic objects in the study of measure rigidity.

ENTROPY FUNCTION FOR AN ACTION α ***OF*** $\mathbb{Z}_+^k, \mathbb{Z}^k$ ***OR*** $\mathbb{R}^k$***.***

- *Entropy function:*

$$h(t) =: h_\mu(t) =: h_\mu(\alpha(t)).$$

- *Ledrappier–Young entropy formula* F. Ledrappier and L.-S. Young, *Ann. of Math.* **122** (1985), 540–574.

$$h_\mu = \sum d_i \chi_i^+ \ (0 \le d_i \le 1)$$

μ an ergodic invariant measure of a C^2 diffeomorphism f.

χ_i the Lyapunov characteristic exponents of f with respect to μ;

d_i *geometric* characteristics of invariant foliations and conditional measures.

PROPERTIES OF THE ENTROPY FUNCTION

From Ledrappier–Young entropy formula:

- $h(\lambda t) = |\lambda| \cdot h(t)$;
- $h(t+s) \le h(t) + h(s)$;
- h is linear in each Weyl chamber.

MEASURE RIGIDITY FOR THE WEYL CHAMBER FLOW ON $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$

Conjecture 0.1 (Margulis) *Let μ be an D-invariant and ergodic probability measure on $X = SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ for $k \geq 3$. Then μ is algebraic, i.e. there is a closed, connected group $L \supset D$ so that μ is the L invariant measure on a single, closed L orbit.*

This conjecture is a special case of much more general conjectures in this direction by Margulis and by A. Katok and R. Spatzier [KS96].

In [EKL03] we prove a weaker version of Margulis conjecture:

Theorem 0.2 (M.Einsiedler, A.K., E.Lindenstrauss) *Let $X = SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. Let μ be an D-invariant and ergodic measure on X. Assume that there is some one parameter subgroup of D which acts on X with positive entropy. Then μ is algebraic.*

HISTORY

The first results for measure rigidity for higher rank hyperbolic actions deal with the Furstenberg $\times m$, $\times n$ problem (the first basic example): After a weaker result of R. Lyons, Rudolph [Ru90] and A. Johnson proved that

any probability measure invariant and ergodic under the action of the semigroup generated by $\times m$, $\times n$ (m, n not powers of the same integer), such that some element of this semigroup acts with positive entropy, is Lebesgue.

When Rudolph's result appeared, I suggested another test model for the measure rigidity: two commuting hyperbolic automorphisms of the three–dimensional torus (the second basic example above) and started to look for a more geometric approach since Rudolph's proof seemed, at least superficially, too closely related to symbolic dynamics.

As a result by 1992 we (Spatzier and A.K.) developed a technique based on the consideration of conditional measures on various invariant foliations, namely stable and unstable foliations of various elements of the actions and their intersections, such as Lyapunov foliations [KaK99].

The method was presented in [KS96] and it is still the foundation of essentially all work for measure rigidity for hyperbolic and partially hyperbolic actions of higher rank abelian groups.

In retrospect, Rudolph's proof can also be interpreted in these terms.

INVARIANCE OF CONDITIONAL MEASURES

The method is based on the observation that if

a nontrivial element a of the action acts on a certain Lyapunov foliation as an isometry then almost every conditional measure is invariant under a certain isomety.

Hence, if a typical point asymptotically returns near another point on the same leaf the conditional measure must be invariant under a translation.

THE ENTROPY RESTRICTION

For a zero entropy measure the conditional measures on stable and unstable foliations (and hence on Lyapunov foliations) are δ–measures. So there are no such returns and hence:

The method of [KS96] and its later elaborations are fundamentally restricted to invariant measures which have *positive entropy* with respect to at least *some elements of the action.*

ERGODICITY (RECURRENCE) RESTRICTION

There is another restriction which looks technical on the surface, but which precludes immediate applications of the results from [KS96] to some of the most interesting cases.

In order to guarantee enough nontrivial returns, ergodic components of the singular element a have to contain leaves of the corresponding Lyapunov foliation.

This does not follow from the ergodicity of a measure with respect to the *whole action*. Thus principal results of [KS96] corrected and and further elaborated in [KaK99] and [KaS02] are coached in such terms as to provide for nontrivial returns.

OVERCOMING THE ERGODICITY RESTRICTION. I: TNS

There are certain cases where the linear algebra of Lyapunov exponents guarantee ergodicity of one–parameter subgroups.

This happens for the second basic example, and more generally, for *Cartan actions* on the torus. Even more generally such the TNS condition is sufficient: (*totally nonsymplectic*; no two Lyapunov exponents are negatively proportional).

OVERCOMING THE ERGODICITY RESTRICTION.II: NON-COMMUTATIVITY OG FOLIATIONS

However, those assumptions are not satisfied for Weyl chamber flows because of the symmetry of the root system or, more geometrically, transverse symplectic structure. In this case all Lyapunov exponents appear in $\pm$ pairs.

At the end of [KS96] there is a brief indication of how the ergodicity problem could be partially overcome for Weyl chamber flows using non-communitativity of the above mentioned foliations.

A far reaching development of this idea together with the observation that the conditional measures of the unstable foliation has the structure of a product measure led to the paper [EK02]. In particular it is shown there that

if μ is an A-invariant measure on $X = SL(k, \mathbb{R})/\Gamma$, and if the entropies of μ with respect to *all* one parameter groups are positive, then μ is the Haar measure.

This is true not only for $\Gamma = SL(k, \mathbb{Z})$ but for every discrete subgroup Γ. There are similar results for Weyl chamber flows on other split simple Lie groups.

An extension to the nonsplit and p-adic cases is in [EK04].

As indicated in [EK02] this result already has implications for the Littlewood conjecture; specifically it implies that

the Hausdorff dimension of the set NL of exceptional points is no greater than one.

THE HIGH ENTROPY CASE

The principal new observation for the split case where all Lyapunov foliations are *one–dimensional*:

If conditional measure in two root directions (Lyapunov foliations) are nontrivial then the conditional measure in the direction of their commutator is Haar.

This allows to treat the *high entropy case*.

In particular for the Weyl chamber flow on $SL(3,\mathbb{R}/\Gamma)$ the only possible non-algebraic measures with positive directional entropy are those which have nontrivial conditionals only for a single pair of the Lyapunov directions (the *low entropy case)*

THE LOW ENTROPY CASE

To deal with this case and hence to come down from one to zero in Hausdorff dimension statement another, quite different approach to measure rigidity was needed. This was developed by E. Lindenstrauss in Invariant measures and arithmetic quantum unique ergodicity, preprint (2003). A special case of the main theorem of that paper is a version of Theorem 0.2 for the Cartan (positive diagonal) action on $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\Gamma$, where Γ is an irreducible lattice in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. The dynamical statement was used by Lindenstrauss to establish quantum unique ergodicity in some arithmetic cases.

TRANSLATION INVARIANCE IN THE LOW ENTROPY CASE

We use an adaptation of the Lindenstrauss method for $SL(n, \mathbb{R})$. It is based on studying the behavior of μ along certain unipotent trajectories, using techniques introduced by Ratner to study invariant measures unipotent flows, in particular the H-property.

We apply this ideas to a conditional measure μ on a Lyapunov foliation which is a priori not even quasi invariant under the corresponding (or any other) unipotent flow.

COMPLEMENTARITY OH THE HIGH AND LOW ENTROPY CASES

In showing that the high entropy and low entropy cases are complementary we use a variant on the Ledrappier-Young entropy formula.

$$h_\mu = \sum d_i \chi_i^+ \ (0 \le d_i \le 1)$$

Notice that such use (which I suggested) is one of the simplifying ideas in G. Tomanov and Margulis' alternative proof of Ratner's theorem.