Lectures on Surfaces: (almost) everything you wanted to know about them

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Topographic data for illustration of earth in Figure 1.12 is taken from the National Oceanic and Atmospheric Administration's ETOPO2v2 data set, available at www.ngdc.noaa.gov.

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Preface

This book is a result of the MASS course in geometry in the Fall semester 2007. MASS core courses are traditionally labeled as analysis, algebra, and geometry, but the understanding of each area is broad, e.g. number theory and combinatorics are allowed as algebra courses, topology is considered as a part of geometry, and dynamical systems as part of analysis. No less importantly, an interaction of ideas and concepts from different areas of mathematics is highly valued.

The topic came to me as very natural under these conditions. Surfaces are among the most common and easily visualized mathematical objects, and their study brings into focus fundamental ideas, concepts, and methods from geometry proper, topology, complex analysis, Morse theory, group theory, and suchlike. At the same time, many of those notions appear in a technically simplified and more graphic form than in their general "natural" settings. So, here was an opportunity to acquaint a group of bright and motivated undergraduates with a wealth of concepts and ideas, many of which would be difficult for them to absorb if presented in a traditional fashion. This is the central idea of the course and the book reflects it closely.

The first, primarily expository, chapter introduces many (but not all) principal actors, such as the round sphere, flat torus, Möbius strip, Klein bottle, elliptic plane, and so on, as well as various methods of describing surfaces, beginning with the traditional representation by equations in three-dimensional space, proceeding to parametric representation, and introducing the less intuitive, but central for our purposes, representation as factor-spaces. It also includes a preliminary discussion of the metric geometry of surfaces. Subsequent chapters introduce fundamental mathematical structures: topology, combinatorial (piecewise-linear) structure, smooth structure, Riemannian metric, and complex structure in the specific context of surfaces. The assumed background is the standard calculus sequence, some linear algebra, and rudiments of ODE and real analysis. All notions are introduced and discussed, and virtually all results proved, based on this background.

The focal point of the book is the Euler characteristic, which appears in many different guises and ties together concepts from combinatorics, algebraic topology, Morse theory, ODE, and Riemannian geometry. The repeated appearance of the Euler characteristic provides both a unifying theme and a powerful illustration of the notion of an invariant in all those theories.

A further idea of both the motivations and the material presented in the book may be found in the Table of Contents, which is quite detailed.

My plan for teaching the course was somewhat bold and ambitious, and could have easily miscarried had I not been blessed with a teaching assistant who became the book's co-author. I decided to use no text either for my own preparations or as a prop for students. Instead, I decided to present the material the way I understand it, with not only descriptions and examples, but also proofs, coming directly from my head. A mitigating factor was that, although sufficiently broadly educated, I am not a professional topologist or geometer. Hence, the stuff I had ready in my head or could easily reconstruct should not have been too obscure or overly challenging.

So, this is how the book came about. I prepared each lecture (usually without or with minimal written notes), and my TA, the third year Ph.D. student Vaughn Climenhaga, took notes and within 24 hours, usually less, prepared a very faithful and occasionally even somewhat embellished version typed in TeX. I usually did some very light editing before posting each installation for the students. Thus, the students had the text growing in front of their eyes in real time.

By the end of the Fall semester the notes were complete: additional work involved further editing and, in a few cases, completing and expanding proofs; a slight reordering of material to make each chapter consist of complete lectures; and in a couple of cases, merging two lectures into one, if in class a considerable repetition appeared. But otherwise the book fully retained the structure of the original one-semester course, and its expansion is due to the addition of a large number of pictures, a number of exercises (some were originally given in separate homework sets, others added later), and some "prose", i.e. discussions and informal explanations. All results presented in the book appeared in the course, and, as I said before, only in a few cases did proofs need to be polished or completed.

Aside from creating the original notes, my co-author Vaughn Climenhaga participated on equal terms in the editorial process, and, very importantly, he produced practically all of the pictures, including dozens of beautiful 3-dimensional images for which, in many cases, even the concept was solely his. Without him, I am absolutely sure that I would not have been able to turn my course into a book in anything approaching the present timeframe, and even if I did at all, the quality of the final product would have been considerably lower.

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